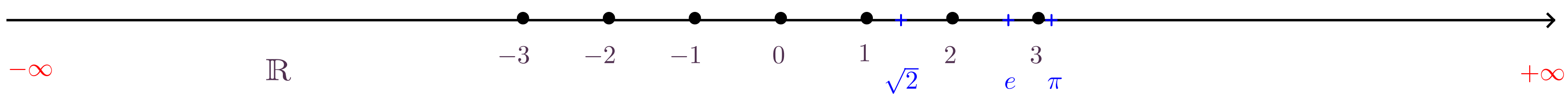
- Calculus is the study of *continuous* change, based upon the fundamental concepts of:

Real numbers $\mathbb{R}$	Functions $f: D \to C$	Function limits $\lim_{x\to c} f(x)$

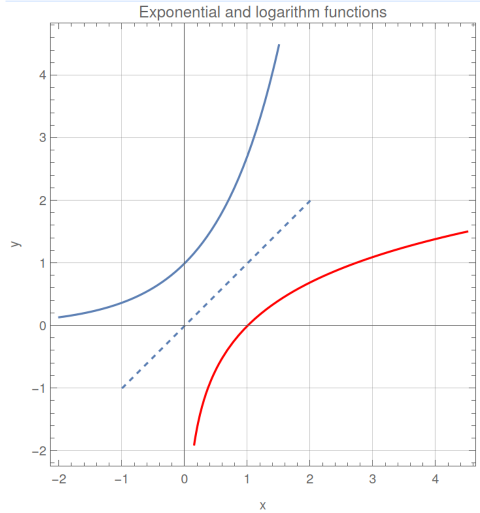
- Real numbers measure continuous quantities and are graphically represented by the *real axis*



- Function f associates to input x from D (domain) *a single* output y from C (codomain)
  $y = f(x)$ states that y is the single output of the function f for given input x
  f is *one-to-one* if output y is produced by *a single* input x, in which case $x = f^{-1}(y)$
  $f^{-1}: C \to D$ is the inverse function of f. Example: $f: \mathbb{R} \to [0, \infty)$, $f(x) = e^x$, $f^{-1}(x) = \ln(x)$
  $f(c)$ is the value of the function *at a single point* c
- The limit $\lim_{x\to c} f(x)$ describes the function f *at an infinity of points* near to c.
  Intuitive definition: $L = \lim_{x\to c} f(x)$ means that $f(x)$ gets closer to L as x gets closer to c
  Formal definition: For any $\varepsilon > 0$ there exists a $\delta(\varepsilon)$ such that from $|x - c| < \delta(\varepsilon)$ it follows that $|f(x) - L| < \varepsilon$.

- A sentence is a complete unit of thought, and contains a subject, a verb, perhaps an object
  Sentences: *Jane read a book*. *Eagles fly*. *Pandora opened the box*. *Have you eaten lunch?*
  Not sentences: *Jane*. *Flew*. *Box*. *Lunch*.
- Subjects are often nouns, either singular or plural: *The boy ran*. *The boys ran*.
- Mathematics statements are similar to English sentences, in that there must be a subject and a verb, perhaps an object. Also, and most importantly, mathematics statements must be either true (T) or false (F).
  Math statements: *Roses are flowers* (T). *All roses are red* (F). $1/2 = 2/4$ (T).
  Not math statements: *1/2*. =. *What is* x*?* $f'(x)$.
- Math statement subjects refer to a single entity (*constants*) or multiple entities (*variables*):
  $1 \cdot 0 = 0$ (T), *Four is even* (T). 1=2 (F). *Any number times zero equals zero*. $\forall x, x \cdot 0 = 0$.
- Some mathematics verbs: $=$ *is equal to* | $\Rightarrow$ *implies* | $\Leftrightarrow$ *is equivalent to* | $\in$ *is in set*
  Examples: $x^2 = 1 \Rightarrow x = -1 \text{ or } x = 1$. "$x^2$ equals 1 implies x equal to 1 or x equal to -1"
  $a = b = c$. "a is equal to b which is equal to c"
- *Always form complete mathematical statements that can be determined to be true or false.*

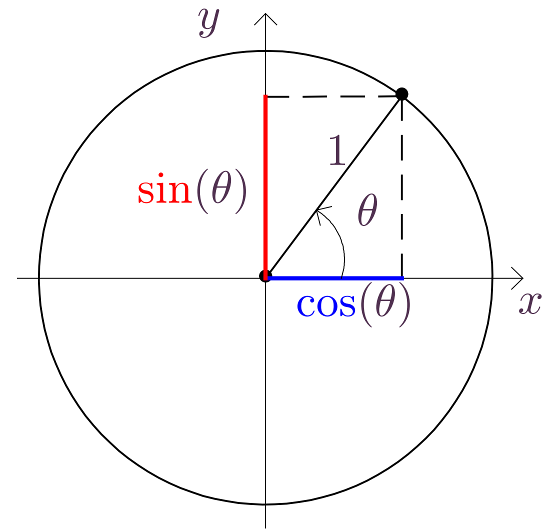
Exponential functions



$e^{\ln(x)} = x$	$\ln e^x = x$
$e^x e^y = e^{x+y}$	$\ln(xy) = \ln(x) + \ln(y)$
$e^{xy} = (e^x)^y$	$\ln(x^y) = y \ln(x)$

$e^0 = 1$	$e \cong 2.72$	$\ln(1) = 0$	$\ln(e) = 1$

Trigonometric functions



$\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$
$\sec(\theta) = \dfrac{1}{\cos(\theta)}$	$\csc(\theta) = \dfrac{1}{\sin(\theta)}$

$\pi \cong 3.14$	$30° \leftrightarrow \dfrac{\pi}{6}$	$45° \leftrightarrow \dfrac{\pi}{4}$	$60° \leftrightarrow \dfrac{\pi}{3}$

$\sin(0) = 0$	$\sin(\pi/6) = 1/2$	$\sin(\pi/4) = \sqrt{2}/2$	$\sin(\pi/3) = \sqrt{3}/2$	$\sin(\pi/2) = 1$
$\cos(0) = 1$	$\cos(\pi/6) = \sqrt{3}/2$	$\cos(\pi/4) = \sqrt{2}/2$	$\cos(\pi/3) = 1/2$	$\cos(\pi/2) = 0$

- Polynomials: $p(t) = a_0 + a_1 t + \cdots + a_n t^n$. Rational functions $r(t) = \frac{p(t)}{q(t)}$, p, q polynomials

Limit from left	$x \to c, x < c$	$\lim_{x \to c^-} f(x) = L_-$
Limit from right	$x \to c, x > c$	$\lim_{x \to c^+} f(x) = L_+$

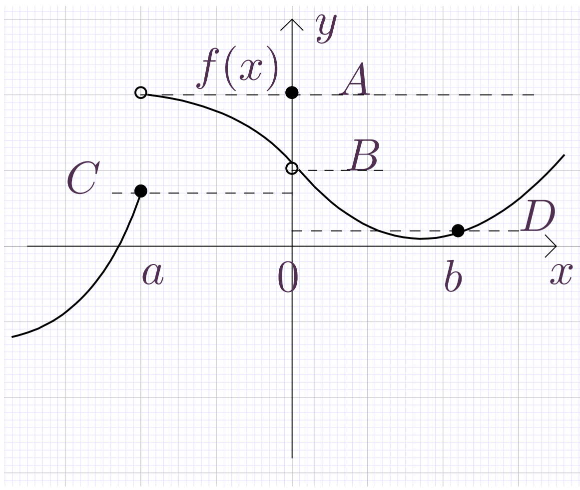
$$\lim_{x \to c} f(x) = L \Leftrightarrow L_- = L_+ = L$$

If $\lim_{x\to c} f(x) = L$, $\lim_{x \to c} g(x) = M$, $a \in \mathbb{R}$ constant, $n \in \mathbb{N}$ a constant, $n \neq 0$, then

$\lim_{x\to c} (f(x) + g(x)) = L + M$	$\lim_{x\to c} (f(x) g(x)) = L M$	$\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M} (M \neq 0)$
$\lim_{x\to c} (a f(x)) = a L$	$\lim_{x\to c} (f(x))^n = L^n$	$\lim_{x\to c} (f(x))^{1/n} = L^{1/n} (f(x) > 0)$

- Limits allow defintion of continuity: $f: D \to C$ continuous at c if $\lim_{x\to c} f(x) = f(c)$

Graphical representations



Is f	continuous?
$\lim_{x\to a^-} f(x) = C$	No.
$\lim_{x\to a^+} f(x) = A$	No.
$\lim_{x\to 0} f(x) = B$	No. $f(0) = A$
$\lim_{x\to b} f(x) = D$	Yes.

Limit computation

- Direct substitution
  $\lim_{x\to 1} x^2 = 1$
- Algebra identities
  $$\lim_{x\to 1} \frac{x^2-1}{x+1} = \lim_{x \to 1} (x-1) = 0$$
  $$\lim_{x\to\infty} \frac{x^2+1}{2x^2+x} = \lim_{x\to 1} \frac{1+\frac{1}{x^2}}{2+\frac{1}{x}} = \frac{1}{2}$$
- l'Hôpital (use derivatives)

- Limits allow evaluation of a ratio of two infinitesimal quantities $\to$ *the derivative*.
  Example: $s(t)$ distance as a function of time, $v(t)=s'(t)$ is the instantaneous velocity
  Notations: $y'(x)=\frac{\mathrm{d}y}{\mathrm{d}x}(x)=\frac{\mathrm{d}}{\mathrm{d}x}y(x)$ values of y' at x, y' is the derivative of function y.

$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$	$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{0}$	$v(t)=\frac{\mathrm{d}s}{\mathrm{d}t}(t)$	s dependent var. t independent var.

- Derivative rules for: addition, product, quotient, composition (*the chain rule*)

$(f+g)'=f'+g'$	$(fg)'=f'g+fg'$	$\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$	$h(x)=f(g(x)), u=g(x)$ $h'(x)=f'(u)\,g'(x)$

- Derivatives table (to be memorized), $a\in\mathbb{R}$, constant, $b\in\mathbb{R}$ a positive constant, $b>0$

$f(x)$	a	x^a	e^x	b^x	$\ln x$	$\log_b x$	$\sin(x)$	$\cos(x)$	$\tan(x)$	$\cot(x)$
$f'(x)$	0	$a\,x^{a-1}$	e^x	$b^x\ln(b)$	$\frac{1}{x}$	$\frac{1}{x\ln b}$	$\cos(x)$	$-\sin(x)$	$\sec^2(x)$	$-\csc^2(x)$

- Derivatives of inverse functions $f\colon D\to C$, $f^{-1}\colon C\to D$, $x\in D$, $y\in C$, $y=f(x)$, $x=f^{-1}(y)$

$\frac{\mathrm{d}y}{\mathrm{d}x}(x)=\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}(y)}$	$y(x)=\sin^{-1}(x)$ $x(y)=\sin(y)$	$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1}(x))=\frac{1}{\frac{\mathrm{d}}{\mathrm{d}y}(\sin(y))}=\frac{1}{\cos(y)}=\frac{1}{\sqrt{1-x^2}}$

$f(x)$	$\sin^{-1}(x)$	$\cos^{-1}(x)$	$\tan^{-1}(x)$	$\cot^{-1}(x)$
$f'(x)$	$\frac{1}{\sqrt{1-x^2}}$	$-\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$	$-\frac{1}{1+x^2}$

Derivative techniques, $y(x)$, y dep. var., x ind. var.

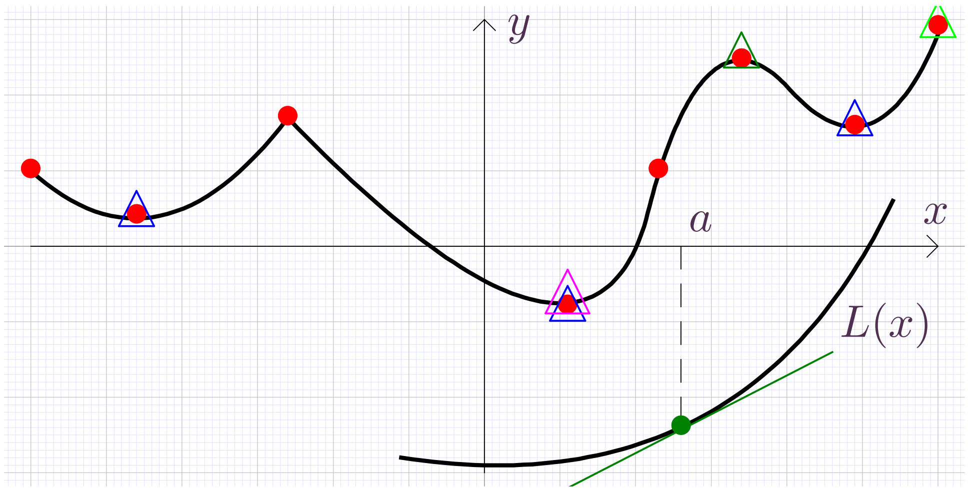
Implicit differentiation: $e^y+x^2=0 \quad e^y y'+2x=0$

Log differentiation: $y(x)=(x-1)^{2/3}x^{1/5}$

$\ln y=\frac{2}{3}\ln(x-1)+\frac{1}{5}\ln(x)$, $\frac{y'}{y}=\frac{2}{3(x-1)}+\frac{1}{5x}$

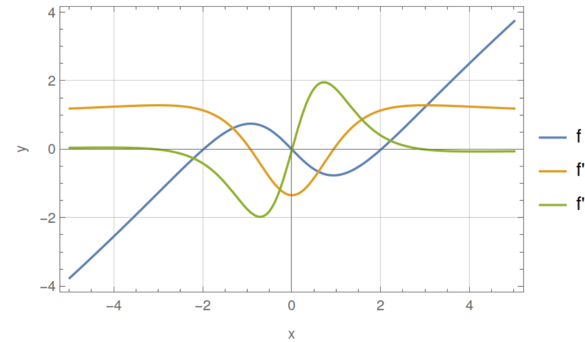
Function extrema (minimum/maximum) $f\colon [a,b] \to \mathbb{R}$

Critical point c: either $f'(c)=0$ or f' does not exist at c



Critical points. Local min ($f''>0$). Local max ($f''<0$). Inflection point ($f''=0$, f' changes sign). Absolute min (at a local min). Absolute max (at a critical point that is not a local max).

Function plots $f\colon \mathbb{R} \to \mathbb{R}$



x	$-\infty$	-2	-0.93	0	0.93	2	∞
f	$-\infty$	0		0		0	∞
f'	$+$	$+$	0	$-$	0	$+$	$+$
	$\nearrow$	$\nearrow$		$\searrow$		$\nearrow$	$\nearrow$
f''	0	$-$	$-$	0	$+$	$+$	0
		$\frown$	$\frown$		$\smile$	$\smile$	

Root. Critical point. Inflection point. Increasing. Decreasing. Concave up, down.

Linear approximation.

Near point $(a, y(a))$, $y(x) \cong L(x)$, $L(x) = y'(a)(x-a) + y(a)$. $y(x+h) \cong L(x+h)$

Differentials. $y = f(x) \Rightarrow \mathrm{d}y = f'(x)\,\mathrm{d}x$

Small increment. $y(x) - y(a) \cong f'(a)\,(x-a)$

Related rates.

1) Define notation. Respect problem notation. 2) Identify independent/dependent variable. 3) Carry out calculus operations, typically: find the critical points of dependent variable.

Limit evaluation (l'Hôpital)

If $\lim_{x\to c} f(x) = 0$, $\lim_{x\to c} g(x) = 0$, f', g' exist

$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$ "0/0" indeterminancy

$\frac{\infty}{\infty} \to \frac{1/\infty}{1/\infty} \Leftrightarrow \frac{0}{0}$, $0 \cdot \infty \to \frac{0}{1/\infty} \Leftrightarrow \frac{0}{0}$,

$\infty - \infty \to \infty\left(\frac{\infty}{\infty} - 1\right) \Leftrightarrow 0/0$

Antiderivative. $F(x)$ antiderivative of $f(x)$.

$F(x) = \int f(x)\,\mathrm{d}x + C$, $F'(x) = f(x)$.

Integrand. Integration variable. Constant.

Evaluate $\int f(x)\,\mathrm{d}x$ by reading differentiation table in reverse

$f(x)$	$a x^{a-1}$	e^x	$\frac{1}{x}$	$\cos(x)$	$-\sin(x)$	$\sec^2(x)$
$\int f(x)\,\mathrm{d}x$	x^a	e^x	$\ln x$	$\sin(x)$	$\cos(x)$	$\tan(x)$

Area function. $A(x) = \int_a^x f(t)\,\mathrm{d}t$

$A(a) = 0$

$A'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$

$F(x) = \int f(x)\mathrm{d}x = A(x) + C$

Fundamental Theorem of Calculus

$\int_a^b f(t)\,\mathrm{d}t = A(b) - A(a)$

$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a)$

Definite integral. $\int_a^b f(x)\,\mathrm{d}x =$ area for with $a \leqslant x \leqslant b$ from $y = f(x)$ to $y = 0$.

$\int_a^a f(x)\,\mathrm{d}x = 0$	$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
$f(x) = f(-x) \Rightarrow$	$\int_{-a}^a f(x)\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$
$f(x) = -f(x) \Rightarrow$	$\int_{-a}^a f(x)\mathrm{d}x = 0$
$a \leqslant c \leqslant b \Rightarrow \int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$	

The definite integral is the limit of Riemann sums (area under curve evaluated by sum of rectangle areas)

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta\to 0}\sum_{k=1}^n f(x_k^*)(x_k - x_{k-1})$$

$a = x_0 < x_1 < \cdots < x_k < \cdots < x_n = b$. $x_{k-1} \leqslant x_k^* \leqslant x_k$

Techniques.

- Substitution (reverse chain rule):
  $\int f(g(x))\,g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$
  with $u = g(x), \mathrm{d}u = g'(x)\,\mathrm{d}x$.
- Substitution for definite integrals
  $\int_a^b f(g(x))\,g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$
- Variable integration limits
  $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{g(x)} f(t)\,\mathrm{d}t = f(g(x))\,g'(x)$
  $\frac{\mathrm{d}}{\mathrm{d}x}\int_{g(x)}^b f(t)\,\mathrm{d}t = -f(g(x))\,g'(x)$