Thursday, August 25, 2022 9:05 AM

Limit Techniques (cont.)

Recall $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ etc. (see L02 & L03)

Additional limit techniques:

- $\lim_{x\to 3} \dfrac{x^2-2x-3}{x-3}$

Recall $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if individual limits exist & $\lim_{x\to a} g(x) \neq 0$.

Investigate behavior $\dfrac{1}{1}=1,\ \dfrac{1}{0.1}=10,\ \dfrac{1}{0.01}=100,\ \dots,\ \dfrac{1}{10^{-n}}=10^n$

$\lim_{x\to 0^+}\dfrac{1}{x}=\infty$

Additional examples

$\lim_{x\to 3^+}\dfrac{2}{x-3}=\infty$

$\lim_{x\to 3^+}\dfrac{x+2}{x-3}=\infty$

$\lim_{x\to 3}\dfrac{x^2-2x-3}{x-3}$ Can we directly evaluate, replace x with 3

for $x=3$ Numerator $x^2-2x-3 = 9-6-3=0$

Solution of quadratic eq

$p(x) = ax^2+bx+c=0$ is

$$x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Solution of linear equation

$mx+n=0$ is $x=-\dfrac{n}{m}$ $(m\neq 0)$

$p(1) = a+b+c$

$p(2) = 4a+2b+c$

$p(x_1)=0 \qquad p(x_2)=0$

x_1, x_2 are roots

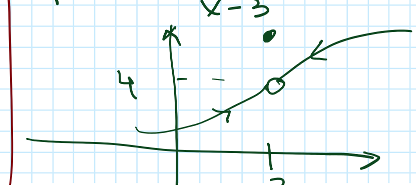
$a(x-x_1)(x-x_2) = p(x)$

$p(x_1) = a\cdot 0\cdot(x_1-x_2)=0$

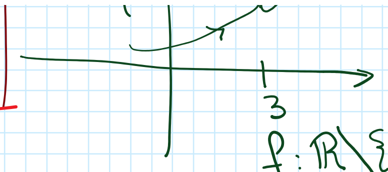
$p(x_2) = a\cdot(x_2-x_1)\cdot 0=0$

Precalc refresher (make sure is always known)

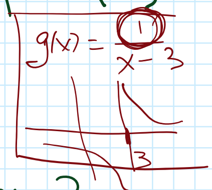
$f(x)=\dfrac{x^2-2x-3}{x-3}$



$$p(x_2) = a\cdot(x_2 - x_1)\cdot 0 = 0$$



$$\lim_{x\to 3} \frac{x^2-2x-3}{x-3} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x+1)}{\cancel{(x-3)}} = \lim_{x\to 3}(x+1) = 4$$



$$x^2-2x-3 = 1\,(x-3)(x-(-1)) = (x-3)(x+1) = x^2-2x-3$$

What did we learn? Algebraic transformation can lead to a limit evaluated directly.

Most often: Algebraic transformation is a factorization

$$a^2-b^2 = (a-b)(a+b)$$

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

$$a^4-b^4 = (a-b)(a^3+a^2b+ab^2+b^3)$$

$$= (a-b)(\underset{3+0=3}{a^3b^0} + \underset{2+1=3}{a^2b} + \underset{1+2=3}{ab^2} + \underset{0+3=3}{a^0b^3})$$

$$a^n-b^n = (a-b)(\underset{\substack{n-1+0=\\n-1}}{a^{n-1}} + \underset{\substack{n-2+1=\\n-1}}{a^{n-2}b} + a^{n-3}b^2 + \dots + ab^{n-2} + b^{n-1})$$

$n \in \mathbb{N}$

a, b can be arbitrary

$a = \sqrt{x} \qquad b = 1$

$$a^2-b^2 = x-1 = (\sqrt{x}-1)(\sqrt{x}+1)$$

- Ex:

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1}\frac{\cancel{\sqrt{x}-1}}{\cancel{(\sqrt{x}-1)}(\sqrt{x}+1)} = \lim_{x\to 1}\frac{1}{\sqrt{x}+1} = \frac{1}{2}$$

- Ex:

$$\lim_{x\to 8}\frac{\sqrt[3]{x}-2}{x-8} \qquad \frac{a-b}{a^3-b^3}$$

taught

Q: Can I directly evaluate?

$x \to 8 \quad x - 8$

thought process

Q1: Can I directly evaluate?

A: It's a quotient. Is denominator non-zero?

N

Q2: Can I algebraically transform?

Can I factor

$\sqrt[3]{x} - 2 \qquad x - 8$

$x - 8 = (\sqrt[3]{x})^3 - 2^3 = (\sqrt[3]{x} - 2)((\sqrt[3]{x})^2 + 2\sqrt[3]{x} + 4)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

$a = \sqrt[3]{x} \qquad b = 2$

$$\lim_{x \to 8} \frac{\sqrt[3]{x} - 2}{x - 8} = \lim_{x \to 8} \frac{\cancel{\sqrt[3]{x} - 2}}{\cancel{(\sqrt[3]{x} - 2)}((\sqrt[3]{x})^2 + 2\sqrt[3]{x} + 4)} = \frac{1}{4+4+4} = \frac{1}{12}$$

$a^3 - b^3 = (\sqrt[3]{x})^3 - 2^3 = x - 8$

$x - 8 \qquad a = \sqrt[3]{x} \quad b = 2$

class Q

Why not $b = -2$, $\quad a = \sqrt[3]{x} \quad b = -2$

$a^3 - b^3 = (\sqrt[3]{x})^3 - (-2)^3 = x - (-8) = x + 8$

No, b must be chosen $b = 2$ for the above example.

More limit techniques

Ex: $\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \cdot x^2$

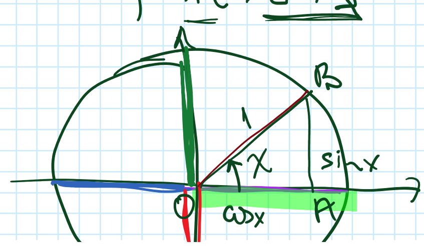
Cannot determine appropriate value for $\sin\left(\frac{1}{x}\right)$ as $x \to 0$.

Meaning

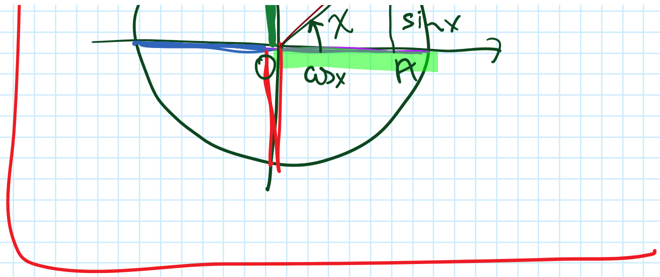
$\lim \sin\left(\frac{1}{x}\right)$ does not

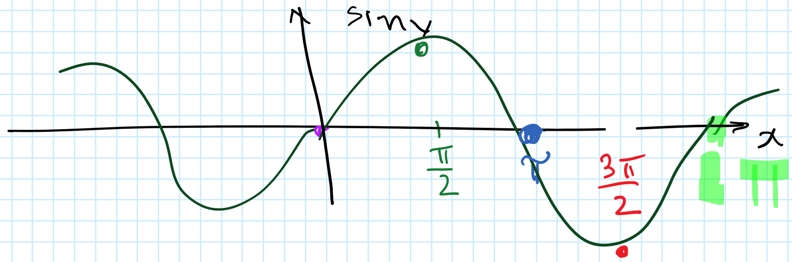
$f(x) = \sin x$ ABSOLUTELY NECESSARY PRECALC KNOWLEDGE

$f: \mathbb{R} \to [-1, 1]$



$\lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist





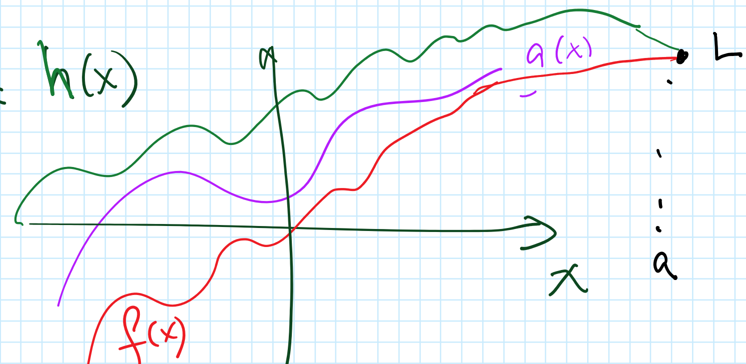
"Squeeze theorem"

Find $\lim\limits_{x \to a} g(x)$ when I know that:

$$f(x) \le g(x) \le h(x)$$

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

Then $\lim\limits_{x \to a} g(x) = L$



Ex: Apply "Squeeze theorem" to

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) x^2 =$$

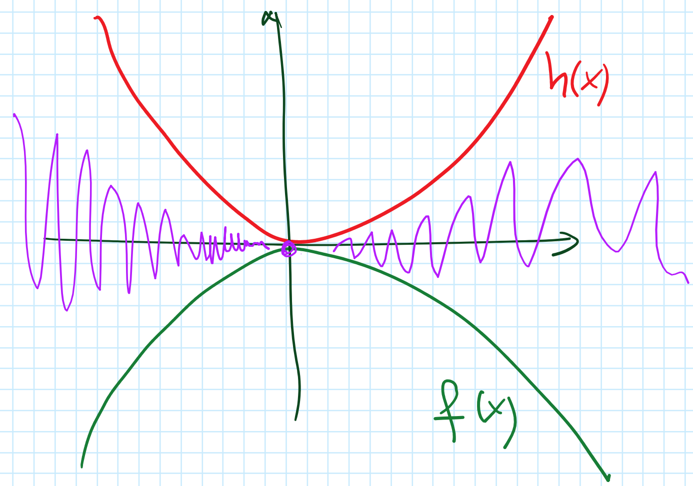
$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

$$(-1)\cdot 2 \le \sin\left(\frac{1}{x}\right)\cdot 2 \le 1\cdot 2$$

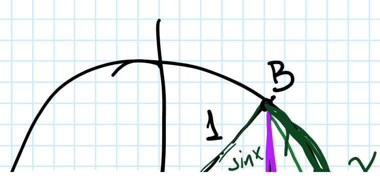
$$(-1)\cdot 3 \le \sin\left(\frac{1}{x}\right) 3 \le 1\cdot 3$$

$$(-1)\cdot x^2 \le \sin\left(\frac{1}{x}\right) x^2 \le 1\cdot x^2$$

$$\begin{array}{ccccc} \| & & & & \| \\ f(x) & \le & g(x) & \le & h(x) \end{array}$$

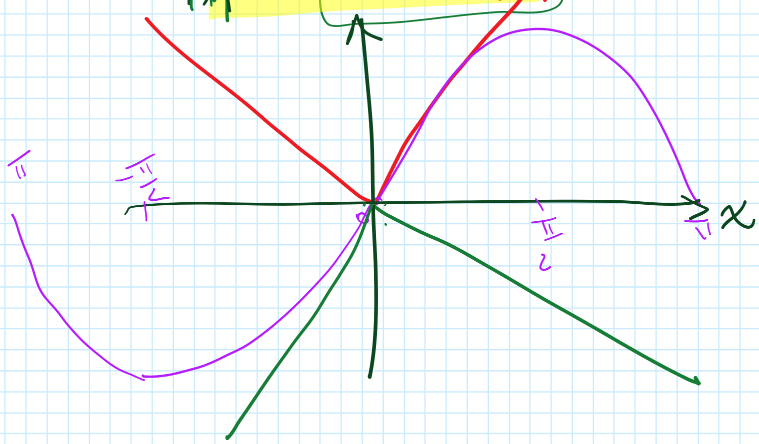


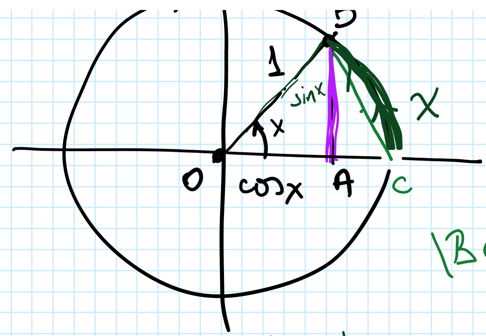
Ex: $\lim\limits_{x \to 0} \sin x = 0$



Ex: $\lim_{x \to 0} \sin x = 0$

$$\boxed{-|x| \le \sin x \le |x|}$$





$|BC| \le x$

Unit circle