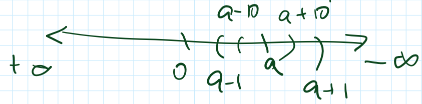
Limits at infinity / Infinite limits

Tuesday, August 30, 2022 9:26 AM

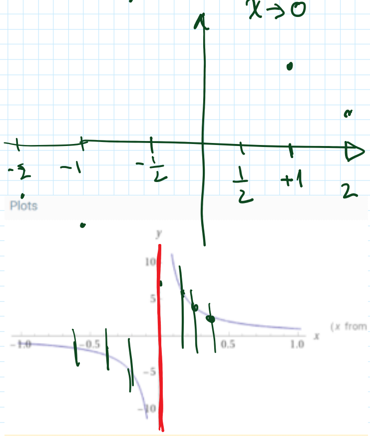
$\lim_{x \to \infty} f(x)$ = "limit of f as $x \to \infty$"
= "Value of f for arbitrarily large x (if it exists)"

$\lim_{x \to a} f(x)$



Vertical asymptotes

$f(x) = \frac{1}{x}$, $\lim_{x \to 0} f(x)$



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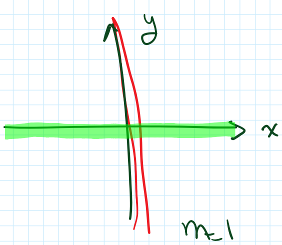
$\nexists \lim_{x \to 0} \frac{1}{x}$ "there does not exist"

$\lim_{x \to 0^+} \frac{1}{x} = \infty$ "The function becomes arbitrarily large"

(Technically, the limit does not exist)

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$



$y = mx + n$ ... , $\left.\begin{matrix} m=0 \\ n=0 \end{matrix}\right\} \Rightarrow y = 0 \cdot x + 0 = 0 + 0 = 0$

$x = 0$ $\qquad y = 0$

$m = 1$
n

$y = 1 \cdot x + n$

$y = mx + n \Rightarrow x = \frac{y-n}{m}$

$y = m \qquad x = \frac{m-n}{m}$

Precalc review

$p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

$f(t) = at + ct^2 + xt^3$

$g(\beta) = \sin \beta x$

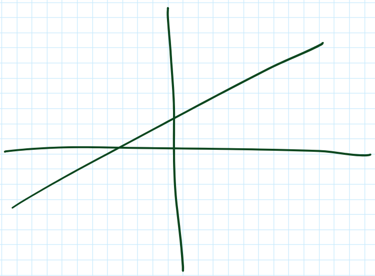
(Difference between "variables" & "parameters")

$x = \frac{y-n}{m} \qquad m = \infty$

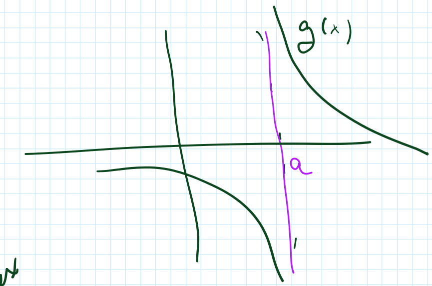
$y = mx + n$

$x = \frac{\cancel{0}}{m}$ $\qquad m = \infty$

$y = mx + n$

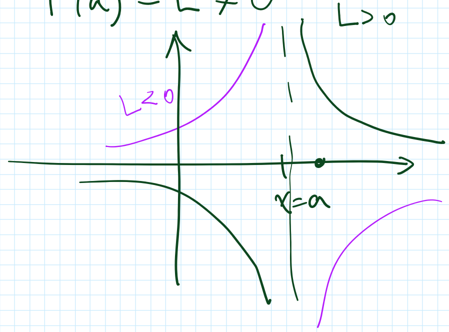


$g(x) = \frac{1}{x-a}$



$h(x) = \frac{F(x)}{x-a}$ can be arbitrarily complex

$F(a) = L \neq 0$ $\quad L > 0$



Exercises

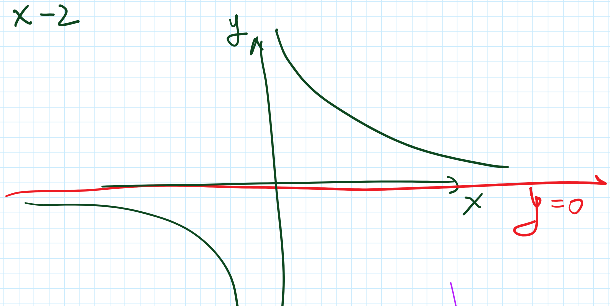
$\lim_{x \to 3} \frac{2-5x}{x-3} \nexists$ $\qquad \lim_{x \to 3^+} \frac{2-5x}{x-3} = -\infty$

$\lim_{x \to 3^-} \frac{2-5x}{x-3} = +\infty$

$\lim_{x \to 2} \frac{\cos(x^2+3x-1)\tan(x)e^{-x} + \sec(x)}{x-2}$

$\lim_{x \to \infty} \frac{1}{x} = 0$

$\lim_{x \to -\infty} \frac{1}{x} = 0$



Horizontal asymptote

$\lim_{x \to \infty} \left(\frac{1}{x} + 1\right) =$

$\lim \left(\frac{1}{x} + 1\right) A = A$

$x \to \infty$

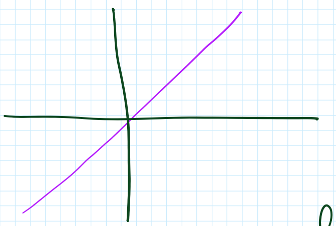
As $x \to \infty$ $\quad \frac{1}{x} \to 0 \quad \left(\frac{1}{x}+1\right)A \to (0+1)A = A$

$$\lim_{x\to\infty} \left(\frac{1}{x}+2\right)F(x) = 2B$$

$$\lim_{x\to\infty} F(x) = B$$

Vertical asymptote with slope $\pm\infty$

Horizontal —"— with slope 0

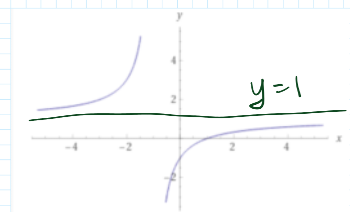


Slanted asymptotes (finite, non-zero slope)

Ex: $f(x) = \dfrac{x-1}{x+1}$

$$\lim_{x\to\infty} \frac{x-1}{x+1} = \lim_{x\to\infty} \frac{\cancel{x}\left(1-\frac{1}{x}\right)}{\cancel{x}\left(1+\frac{1}{x}\right)} =$$

$$= \lim_{x\to\infty} \frac{1-\frac{1}{x}}{1+\frac{1}{x}} = 1$$



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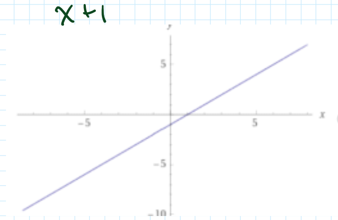
Ex $\quad g(x) = \dfrac{x^2-1}{x+1}$

$$a^2-b^2 = (a-b)(a+b)$$
$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$
$$a^4-b^4 = (a-b)(a^3+a^2b+ab^2+b^3)$$

$$g(x) = \frac{x^2-1}{x+1} =$$

$$g(x) = \frac{(x-1)(x+1)}{x+1} = x-1$$



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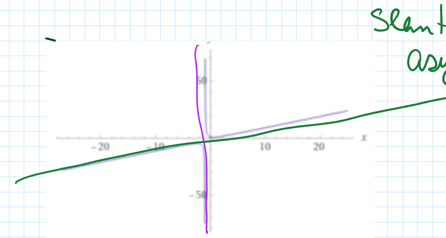
$$\lim_{x\to\infty} g(x) = \infty \qquad \lim_{x\to-\infty} g(x) = -\infty$$

Ex: $\quad h(x) = \dfrac{x^2+1}{x+1}$



$$\lim_{x\to\infty} h(x) = \infty$$

$h(x) \quad P(x) \quad x^2+1 \qquad P(x) = x^2+1$

$x \to \infty$

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$$h(x) = \frac{p(x)}{q(x)} = \frac{x^2+1}{x+1} \qquad p(x) = x^2+1, \quad q(x) = x+1$$

$$q(x)\,\overline{)\,p(x)}^{\;r(x) \,+\, R(x)}$$

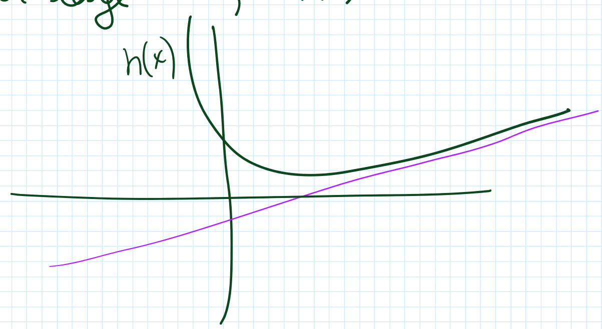
$$h(x) = \frac{x^2+1}{x+1} = \underbrace{x-1}_{\text{quotient}} + \underbrace{\frac{2}{x+1}}_{\text{remainder}}$$

$$\begin{array}{r l}
 & \quad x - 1 \\
x+1 & \overline{)\; x^2 + 0\cdot x + 1} \\
 & \;\; x^2 + x \\
 & \overline{= \quad\; -x + 1} \\
 & \qquad -x - 1 \\
 & \overline{\qquad = \quad 2}
\end{array}$$

(Polynomial long division)
Must be known.

$$\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \left(x - 1 + \frac{2}{x+1} \right)$$

For large x, $h(x)$ behaves like $x-1$

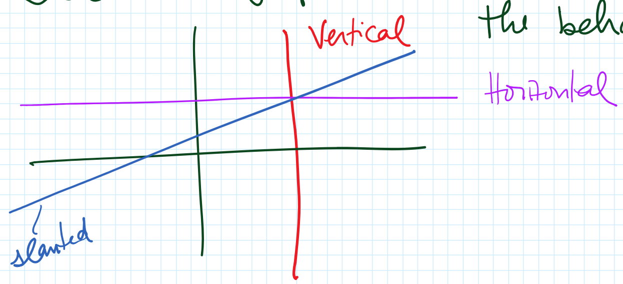


Importance of slanted asymptotes $\Rightarrow$
Allows a simple description of complicated behavior

Ex: $h(x) = \frac{x^2+1}{x+1}$ $\qquad F(x) = h(x) \sin x$

Behaves like $(x-1)\sin x$ for large x

Recap: Asymptotes = "lines that approximate the behavior of a function"

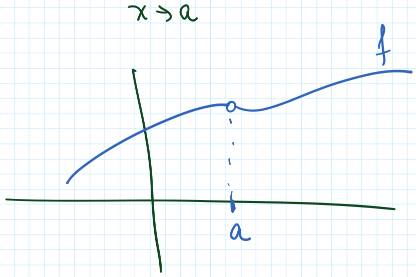


Continuity of functions

def $f:\mathbb{R}\to\mathbb{R}$ is continuous at a if

$\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = f(a)$

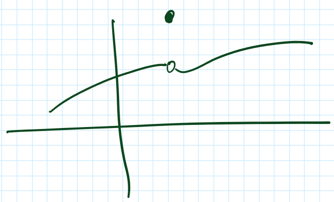
Ex:



Not continuous at a

$\lim_{x\to a} f(x) = L$

Ex:

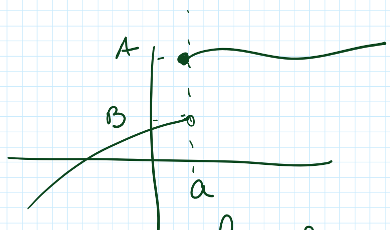


$f(a)$ defined? Yes

$\lim_{x\to a} f(x) = L$

f is not continuous

Ex:

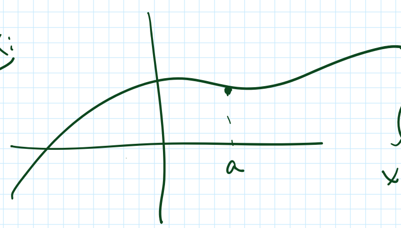


$f(a)$ defined?

$\lim_{x\to a} f(x)$ does not exist

$\lim_{x\to a^-} f(x) = B \qquad \lim_{x\to a^+} f(x) = A$

Ex:



$f(a)$ exists (is defined)

$\lim_{x\to a} f(x) = f(a)$

Ex:

$f(x) = \dfrac{x+1}{x^2-1}$ $\quad$ f is continuous at $x=0$

step 1 check if $f(0)\ \exists$: $\quad f(0) = -1$

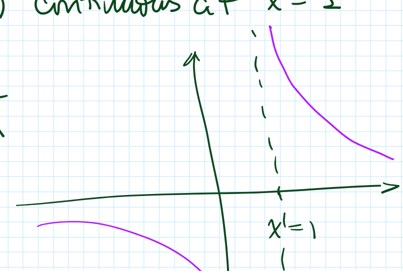
step 2 check if $\lim_{x\to 0} f(x)\ \exists$: $\quad \lim_{x\to 0} f(x) = -1$

step 3 Are they equal? Yes

Conclusion: $\Rightarrow$ $f(x)$ is continuous at $x=0$

Ex $f(x) = \dfrac{x+1}{x^2-1}$ $\quad$ $f(x)$ continuous at $x=1$

$f(x) = \dfrac{x+1}{(x-1)(x+1)} = \dfrac{1}{x-1}$

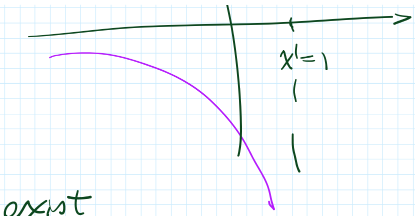


$\lim_{x\to 1^-} f(x) = -\infty$

Not equal $\Rightarrow$

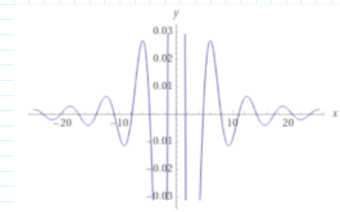
$\lim_{x\to 1^-} f(x) = -\infty$ Not equal ⇒

$\lim_{x\to 1^+} f(x) = +\infty$



$\lim_{x\to 1} f(x)$ does not exist

Ex: $f(\theta) = \frac{\cos\theta}{\theta^2}$



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$f(\theta)$ is even

$f(\theta) = f(-\theta)$

$\theta^2 = (-\theta)^2 = (-1\cdot\theta)^2 = (-1)^2\theta^2 = \theta^2$ ✓

$\cos\theta = \cos(-\theta)$