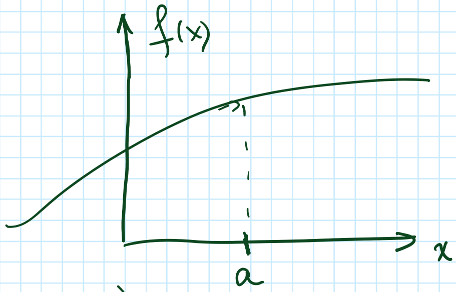
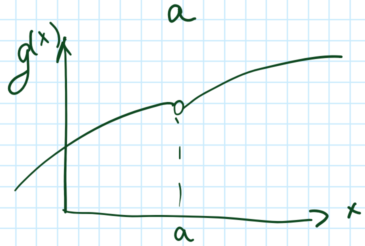


$f: \mathbb{R} \to \mathbb{R}$

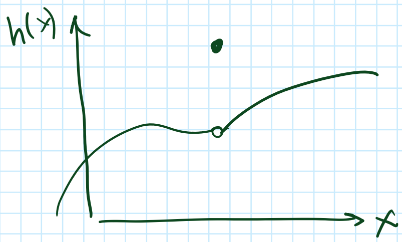
$f(x)$ output

x input

$\lim_{x \to a} f(x) = f(a)$ $\quad f$ is continuous at $x = a$

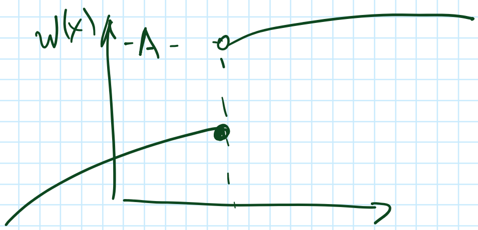


$g(x)$ $\quad g(a)$ undefined



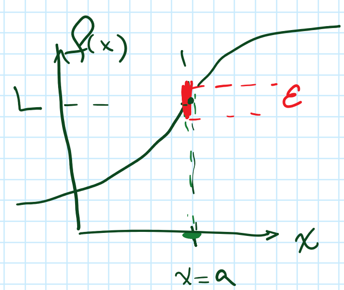
$h(a)$ defined

$h(a) \neq \lim_{x \to a} h(x)$



$\lim_{x \to a^-} w(x) = w(a)$

$\lim_{x \to a^+} w(x) \neq w(a)$

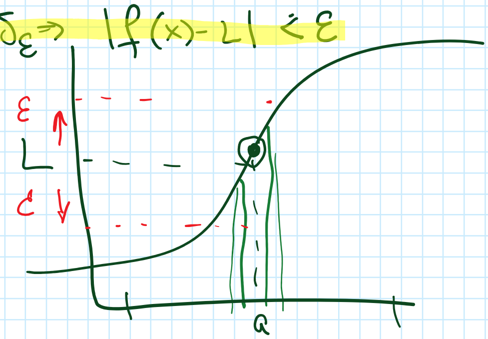


(~~Precise~~) Mathematical definition of a Limit

$f: \mathbb{R} \to \mathbb{R}$ has limit L at $x = a \in \mathbb{R}$ if

$\forall \ \varepsilon > 0 \quad \exists \ \delta_\varepsilon = \delta(\varepsilon)$ such that $|x - a| < \delta_\varepsilon \Rightarrow |f(x) - L| < \varepsilon$

"No matter" "Whatever" $\qquad$ "there must exist"



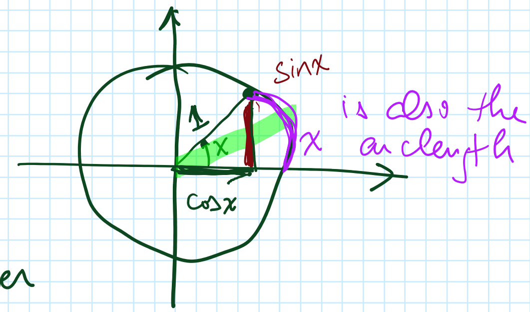
($\exists!$ "There must exist & be unique")

($\alpha \beta \gamma \delta \varepsilon \quad \xi \eta \zeta$)

To apply this definition you must explicitly find $\delta(\varepsilon)$

To apply this definition you must explicitly find $\delta(\varepsilon)$

Ex: $f(x) = \frac{\sin x}{x}$ $\lim_{x\to 0} f(x)$



Observe as $x \to 0$ $\sin x \to 0$

Scenarios

1) sin x goes to zero much faster than x

$$\frac{\sin x}{x} \to 0$$

2) sin x goes to zero much slower than x

$$\frac{\sin x}{x} \to \pm\infty$$

3) sin x goes to zero about the same as x

$$\frac{\sin x}{x} \to \text{some non-zero finite number}$$

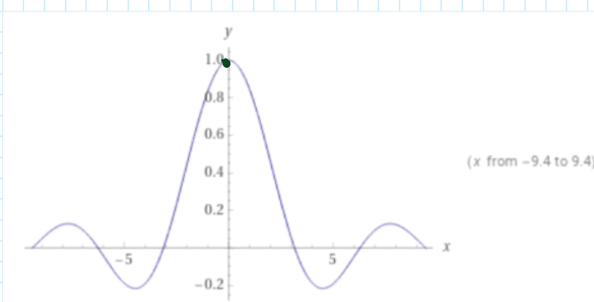
$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$\forall \varepsilon > 0 \quad \exists\, \delta(\varepsilon) = \delta_\varepsilon$ s.t

if $|x - 0| < \delta_\varepsilon$ then

$$\left| \frac{\sin x}{x} - 1 \right| < \varepsilon$$

$$-\varepsilon < \frac{\sin x}{x} - 1 < \varepsilon$$



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Objective: find which values of x enforce the inequality

$$\boxed{1 - \varepsilon < \boxed{\frac{\sin x}{x}} < \varepsilon + 1}$$

$|\sin x| < |x| \quad x \neq 0$

$$\left| \frac{\sin x}{x} \right| < 1$$

$$\Downarrow$$

$$-1 < \boxed{\frac{\sin x}{x}} < 1$$

Precalc refresher

$a < b \Rightarrow \frac{a}{2} < \frac{b}{2}$

$a < b \Rightarrow \frac{a}{c} < \frac{b}{c} \quad c > 0$

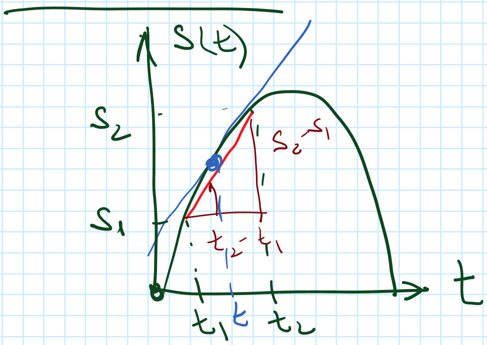
$-1 < \frac{\sin x}{x} < 1$

$a < b \Rightarrow \frac{a}{c} < \frac{b}{c}$

~~$2 < 3$ divide -1 $\Rightarrow -2 < -3$~~

Why did we do this?

- It's not going to be on a test
- It is an introduction to recognition of rigorous arguments



"Velocity" $s(t)$ = distance traversed in time

Ex: $s(t) = Vt - \frac{1}{2} g t^2$

$s(t) = 10t - 5t^2$

"Velocity" $= \frac{\text{distance interval}}{\text{time interval}} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$

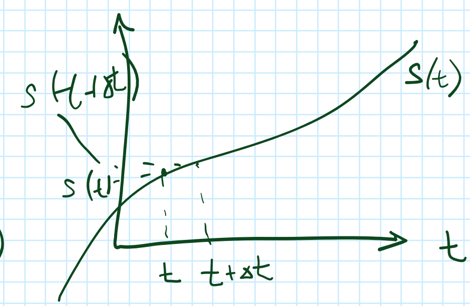
→ Average velocity

def: Instantaneous velocity $v(t)$

If an object's trajectory is $s(t)$

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}$$

Δt = interval of time over which we take 2 positions



$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ A & B & \Gamma & \Delta \end{pmatrix}$$

def: $f: \mathbb{R} \to \mathbb{R}$ has derivative $f'(x_0)$ at $x = x_0$ if the limit $\lim \frac{f(x) - f(x_0)}{}$ exists

$$\lim_{x \to x_0} \quad x - x_0$$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$\Delta x = x - x_0 \qquad x_0 + \Delta x = x$$

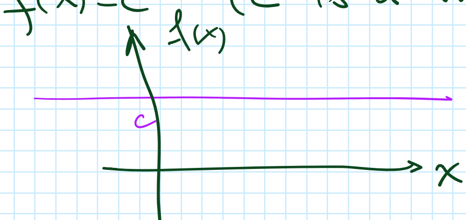
__def__: $f: \mathbb{R} \to \mathbb{R}$ has derivative $f'(x)$ if for $\forall x \in \mathbb{R}$

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ exists}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Derivatives of common functions

$\boxed{0)}$ $f: \mathbb{R} \to \mathbb{R} \qquad f(x) = c$ (c is a constant)



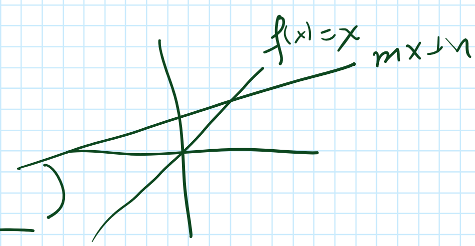
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

$\boxed{1)}$ $f: \mathbb{R} \to \mathbb{R} \qquad f(x) = x \qquad f(1) = 1 \quad f(2) = 2$

~~f("Dustin Hoffman") = "Dustin Hoffman"~~

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x} = 1$$

1.a) $f(x) = mx + n$



$$f'(x) = \lim_{\Delta x \to 0} \frac{(m(x + \Delta x) + n) - (mx + n)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{m \cdot x + m \cdot \Delta x + n - mx - n}{\Delta x} = \lim_{\Delta x \to 0} \frac{m \Delta x}{\Delta x} = m$$

$$= \lim_{\Delta x \to 0} \frac{m \cdot x + m \cdot \Delta x + n - m x - n}{\Delta x} = \lim_{\Delta x \to 0} \frac{m \Delta x}{\Delta x} = m$$

$$f(x) = mx + n = \underbrace{m \cdot g(x)}_{G(x)} + \underbrace{n \cdot h(x)}_{H(x)}$$

$$g(x) = x$$

$$h(x) = 1$$

$$f'(x) =$$

$$G'(x) = \lim_{\Delta x \to 0} \frac{G(x+\Delta x) - G(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{m \cdot g(x+\Delta x) - m \cdot g(x)}{\Delta x}$$

$$= m \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} = m g'(x) = m$$

$$H'(x) = \ldots = 0$$

Rule (linear combination rule)

$$f(x) = a\, g(x) + b\, h(x)$$

function = constant | function + constant | function

$$f'(x) = a\, g'(x) + b\, h'(x)$$

$a^2 - b^2 = (a-b)(a+b)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$\boxed{n}$ $f(x) = x^n \quad n \in \mathbb{N}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$(x+\Delta x)^n - x^n$

$$(x+\Delta x)^n - x^n$$

$$a^n - b^n = (a-b)(\underbrace{a^{n-1}\cdot b^0 + a^{n-2}b^1 + a^{n-3}b^2 + \dots + ab^{n-2} + a^0 b^{n-1}})$$

$$\left.\begin{array}{l} a = x+\Delta x \\ b = x \end{array}\right\} \Rightarrow (a-b) = \Delta x$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}\left[(x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + \dots + x^{n-1}\right]}{\cancel{\Delta x}}$$

limit of a sum

$$= \underbrace{\lim_{\Delta x\to 0}(x+\Delta x)^{n-1} + \lim_{\Delta x \to 0}(x+\Delta x)^{n-2}x}_{n \text{ such limits}} + \dots + \lim_{\Delta x \to 0} x^{n-1}$$

$$\lim_{\Delta x \to 0}(x+\Delta x)^{n-1}\cdot 1 = x^{n-1}$$

$$\lim_{\Delta x \to 0}(x+\Delta x)^{n-2}\cdot x = x^{n-1}$$

$$\lim_{\Delta x \to 0} x^{n-1} = x^{n-1}$$

$f(x) = x^n$ has derivative $f'(x) = n\,x^{n-1}$ Memorize