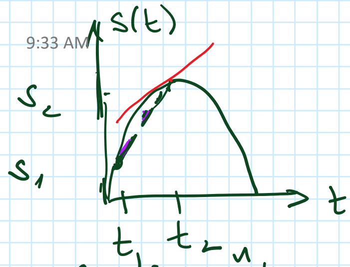
Tuesday, September 13, 2022 9:33 AM

More differentiation rules

Recall



$s(t)$

$s_1 = s(t_1)$; $s_2 = s(t_2)$

Average velocity ("physics") $\Leftrightarrow$ slope ("geometry") of secant

$$s'(t_2) = \lim_{t_1 \to t_2} \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \text{instantaneous velocity ("physics")} \Leftrightarrow \text{slope ("geometry") of tangent}$$

$f: \mathbb{R} \to \mathbb{R}$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$ derivative at a point

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ derivative function

Rules of differentiation

1) $\frac{d}{dx}(c) = 0$ (c a constant)

2) $\frac{d}{dx} x^n = n x^{n-1}$ ($n \in \mathbb{N}$)

3) $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$ ($(f+g)' = f' + g'$ another notation $D(f+g) = Df + Dg$)

4) $\frac{d}{dx}(fg) = f \frac{dg}{dx} + \frac{df}{dx} g$

$$(fg)' = f'g + fg'$$

e^x $e = 2.718$ Basis of natural logarithm

Let's consider $f(x) = 2^x$ $g(x) = 3^x$

(1st Question is $\frac{df}{dx}$ computable by rule 2 $\frac{d}{dx}x^n = n x^{n-1}$?)

Yes ? 6 No ? 11 Absent ~20

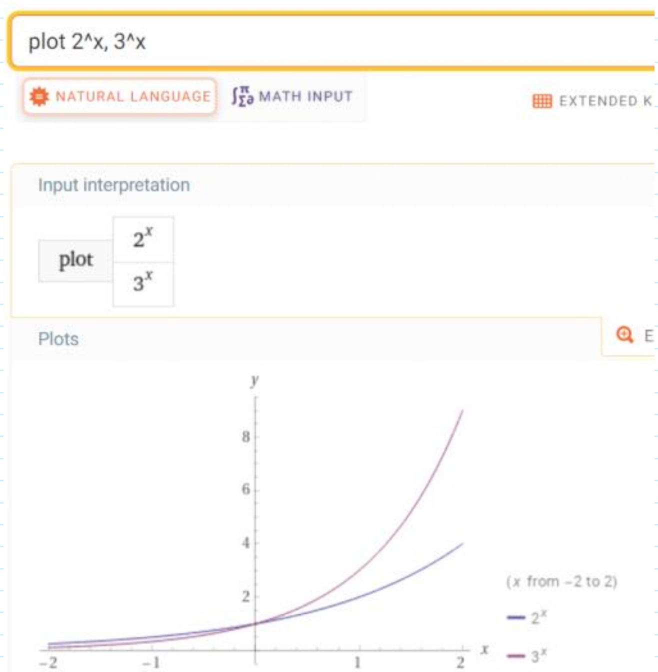
Silent majority "Huh?"

Common mistake

$2^{\boxed{x}}$ Exponent

Base

Numerical experiment



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$$f(x) = 2^x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h} =$$

$$= \lim_{h \to 0} \frac{2^x(2^h - 1)}{h}$$

$$g'(x) = \lim_{h \to 0} \frac{3^x(3^h - 1)}{h}$$

Choose x to gain insight

$$f'(x) = \lim_{h \to 0} \frac{2^x(2^h - 1)}{h} \qquad g'(x) = \lim_{h \to 0} \frac{3^x(3^h - 1)}{h} \quad (*)$$

Choose $x = 0$ to get rid of $2^x, 3^x$ in

$$f'(0) = \lim_{h \to 0} \frac{2^h - 1}{h} \qquad g'(0) = \lim_{h \to 0} \frac{3^h - 1}{h}$$

$$\left(\frac{0}{0}\right) \qquad \left(\frac{0}{0}\right)$$

Screen clipping taken: 9/13/2022 9:56 AM table[(2^h-1)/h,{h,0.01,0.1,0.01}]

NATURAL LANGUAGE MATH INPUT

Assuming "h" is a variable | Use as a u

Input

$\text{Table}\left[\frac{2^h-1}{h}, \{h, 0.01, 0.1, 0.01\}\right]$

Result

h	$\frac{2^h-1}{h}$
0.01	0.695555
0.02	0.697974
0.03	0.700404
0.04	0.702846
0.05	0.705298
0.06	0.707763
0.07	0.710238
0.08	0.712726
0.09	0.715224
0.1	0.717735

h	$\frac{3^h-1}{h}$
0.01	1.10467
0.02	1.11077
0.03	1.11692
0.04	1.12311
0.05	1.12935
0.06	1.13563
0.07	1.14196
0.08	1.14834
0.09	1.15476
0.1	1.16123

$\frac{2^h-1}{h}$ $\quad$ $\frac{e^h-1}{h} \to 1$ $\quad$ $\frac{3^h-1}{h}$

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Numerical experiment suggests $\exists\, e$, $2<e<3$ s.t.

$$\lim_{h\to 0} \frac{e^h-1}{h} = 1$$

Experiment

(Preview exact proof $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$)

Proof (saved for later)

Find derivative

$$h(x) = e^x$$

$$h'(x) = \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h-1)}{h} = e^x \boxed{\lim_{h\to 0} \frac{e^h-1}{h}}$$

$$h \to 0 \qquad h \qquad h \to 0 \qquad h \qquad h \to 0 \quad h$$

$$= e^x$$

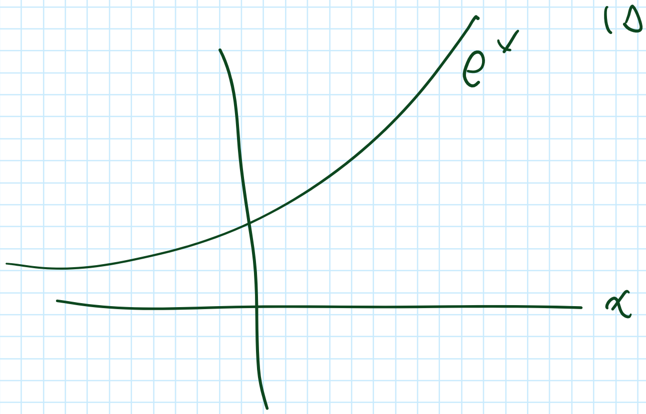
Lo & behold!

Theorem $$(e^x)' = \frac{d}{dx} e^x = e^x$$

0 "neutral element" for $\oplus$ $(x + 0 = x)$

1 " —"— " for $\odot$ $(x \cdot 1 = x)$

$$\frac{d}{dx} e^x = e^x$$ e^x is a very special function, is not changed by differentiation

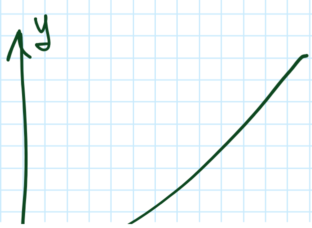


Differentiation of exponentials a^x a = basis x = exponent

$$\frac{d}{dx} e^x = e^x$$ What happens if $a \neq e$

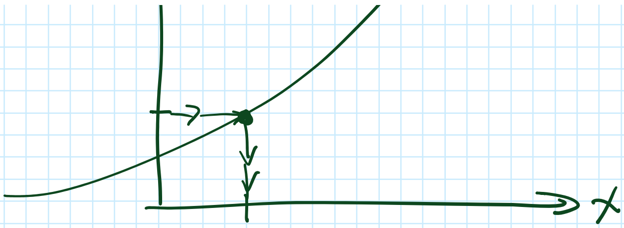
Properties of logarithms (~~Precalc~~ Review) Crucial to know this

= inverse function of exponentiation



a^x $(a \in \mathbb{R}_+ \ a > 1)$

Inverse of $f(x) = a^x$



Inverse of $f(x) = a^x$

$f^{-1}(y) = \log_a y$

$$a^{\log_a x} = x$$

$$\log_a\left(a^{\log_a x}\right) = \log_a x$$

Apply *

$$\log_a x \underbrace{\log_a a}_{=1} = \log_a x$$

$$\boxed{\log_a x = \log_a x}$$

True since $\log_a$ is one-to-one

Operations with logarithms

$\log(ab) = \log a + \log b$

$\log(a\,a) = \log a^2 = 2\log a$

$\log(a\,a\,a) = \log a^3 = 3\log a$

$\log(a^b) = b\log a \quad (*)$

$\log_a b$ To what power must a be raised to obtain b

Derivative of $f(x) = a^x$

$$\boxed{f'(x) = (\ln a)\, a^x} = (\log_e a)\, a^x$$

Extra credit Prove this! (1 C.P.) Up to Sep 27.

$\boxed{\ln a}$ = "natural logarithm of a"

= logarithm in base e

= $\log_e a$

[Reward for those in class (or those that at least read these posted notes]

Derivatives of other common functions

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} a^x = (\ln a)\, a^x$$

Part of differentiation table to be memorized

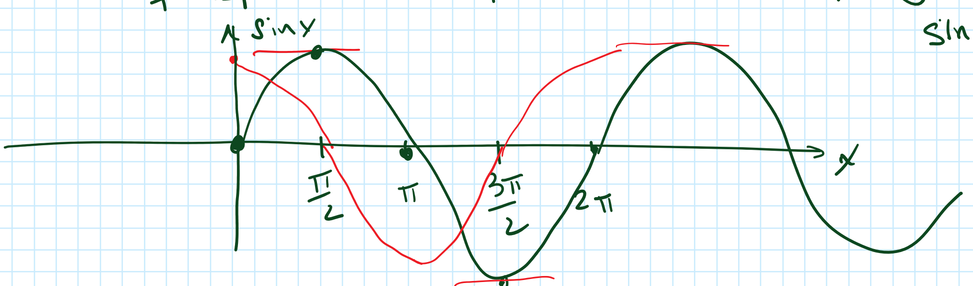
$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

Proof of above formulas "Apply the definition"

$\sin x = \sin(x + 2k\pi)$
$k \in \mathbb{Z}$



x	0		$\frac{\pi}{2}$		π		$\frac{3\pi}{2}$		2π
$\sin x$	0	$\nearrow$	1	$\searrow$	0	$\searrow$	-1	$\nearrow$	0
$(\sin x)'$		$+$	0	$-$		$-$	0	$+$	

$$f(x) = \sin x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

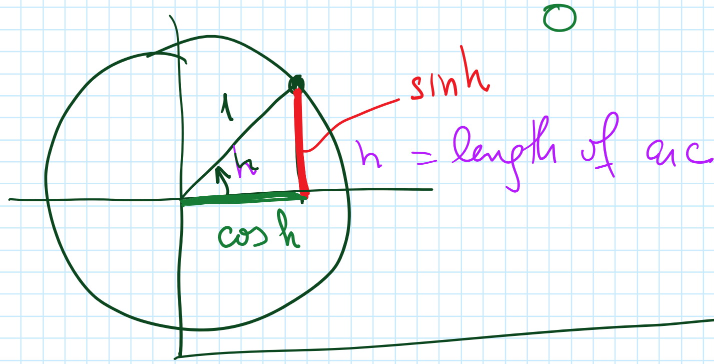
(Precalc: $\sin(a+b) = \sin a \cos b + \sin b \cos a$)

$$f'(x) = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sin(x)(\cos(h) - 1)}{h} + \cos x \frac{\sin h}{h} \right]$$

$$= \lim_{h \to 0} \frac{\sin x[\cos h - 1]}{h} + \lim_{h \to 0} \frac{\sin h}{h} \cos x$$

$$= \sin x \lim_{h \to 0} \underbrace{\frac{\cos h - 1}{h}}_{= 0} + \cos x \lim_{h \to 0} \underbrace{\frac{\sin h}{h}}_{= 1}$$

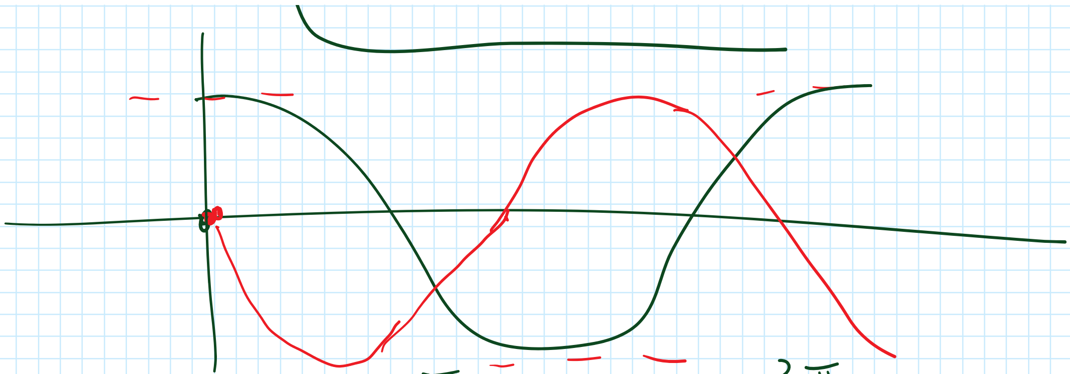


$$\boxed{\frac{d}{dx} \sin x = \cos x}$$

Obs: Q1) Is this a simple calculation? Not really

Hence we memorize.

$$\boxed{\frac{d}{dx} \cos x = - \sin x}$$



x	0		$\frac{\pi}{2}$		π		$\frac{3\pi}{2}$		2π
$\cos x$	1	$\searrow$	0	$\searrow$	-1	$\nearrow$	0	$\nearrow$	1
$(\cos x)'$	0	$-$		$-$	0	$+$		$+$	0