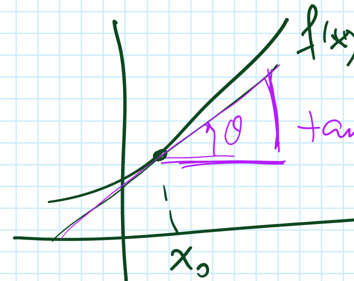
More differentiation rules

Recall: $f'(x_0) = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$



$\tan\theta = \text{slope} = \dfrac{df}{dx}(x_0)$

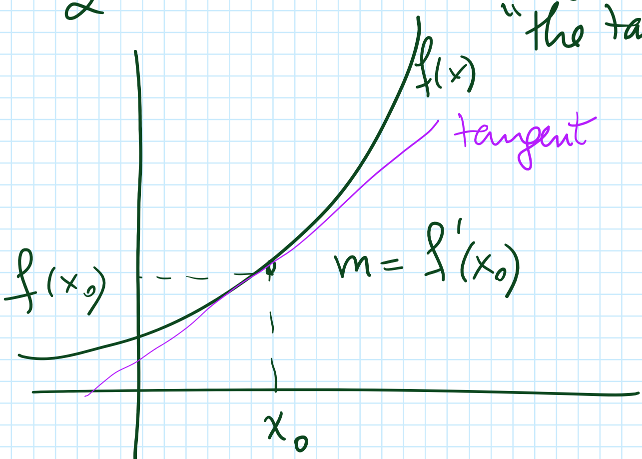
$\dfrac{d}{dx}(c) = 0$; $\dfrac{d}{dx}(x^n) = n\, x^{n-1}$ $n \in \mathbb{N}$

$\dfrac{d}{dx}(a_n x^n + \dots + a_1 x + a_0) = n\, a_n x^{n-1} + \dots + a_1$

$\dfrac{d e^x}{dx} = e^x$; $\dfrac{d a^x}{dx} = \ln a \; a^x$; $\dfrac{d}{dx} \sin x = \cos x$; $\dfrac{d}{dx} \cos x = -\sin x$

$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ (derivative function)

Equation for a ~~the~~ tangent line to a point on graph of $f(x)$
"the tangent"



$m = f'(x_0)$

$f : \mathbb{R} \to \mathbb{R}$

$y = mx + n$

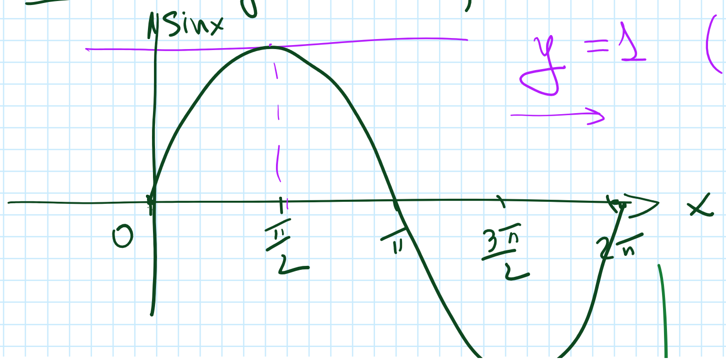
$y = f'(x_0)\, x + f(x_0) - f'(x_0)\, x_0$

$y - f(x_0) = f'(x_0)(x - x_0)$ (*) Eq. for tangent

$f(x_0) = f'(x_0)\, x_0 + n$

$n = f(x_0) - f'(x_0)\, x_0$

Ex: Tangent to $f(x) = \sin x$ at $x_0 = \dfrac{\pi}{2}$



$y = 1$ (by direct observation)

Apply formula (*)

$y - f(x_0) = f'(x_0)(x - x_0)$

$y - f\left(\frac{\pi}{2}\right) = f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)$

$f'(x) = \cos x$

$\cos\frac{\pi}{2} = 0$

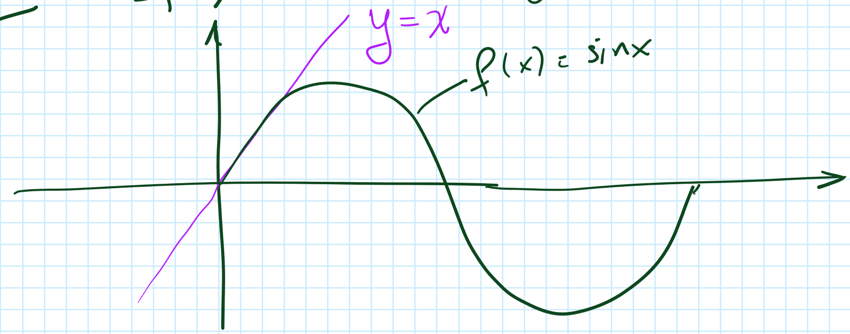
$$y - f\left(\frac{\pi}{2}\right) = f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)$$

$$y - \sin\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)$$

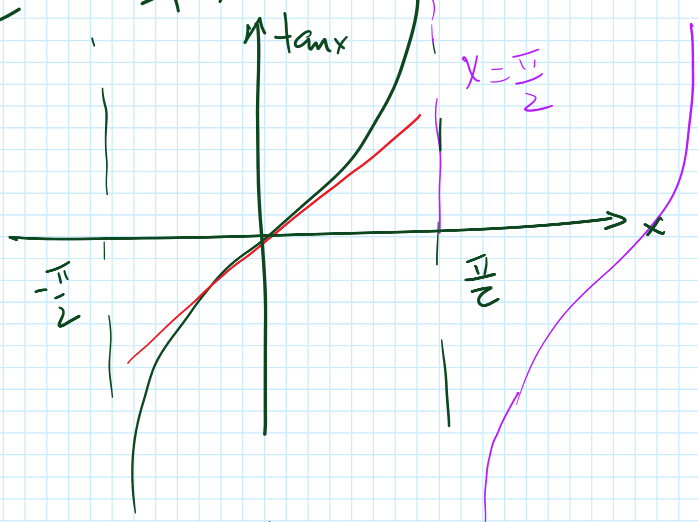
$$y - 1 = 0\cdot\left(x - \frac{\pi}{2}\right)$$

$$y = 1$$

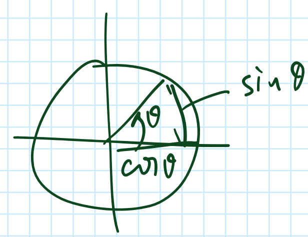
Ex $\quad f(x) = \sin x \qquad x_0 = 0$



Ex $\quad f(x) = \tan x \qquad x_0 = \frac{\pi}{2}$



$f(x) = \tan x = \dfrac{\sin x}{\cos x}$



Ex $\quad f(x) = \tan x \qquad x = 0$

Derivative of quotients $\qquad \frac{a}{b} = a\cdot\left(\frac{1}{b}\right)$

$$f(x) = \frac{P(x)}{Q(x)} \qquad Q(x) \neq 0$$

Rewrite as $\qquad P(x) = f(x)\,Q(x)$

Rewrite as $p(x) = f(x)g(x)$

$$\frac{d}{dx}(p(x)) = \frac{d}{dx}(f(x)g(x)) = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}$$

$$p' = (fg)' = f'g + fg'$$

$$f' = \frac{1}{g}(p' - fg') = \frac{1}{g}\left(p' - \frac{p}{g}g'\right)$$

$$f = \frac{p}{g} \qquad f' = \frac{p'g - pg'}{g^2}; \quad f = pg; \quad f' = p'g + pg'$$

Quotient

<u>Ex</u>: $f(x) = \dfrac{x^2+x+2}{x+1} = \dfrac{p(x)}{g(x)}$ $\qquad p'(x) = 2x+1$, $\quad g'(x) = 1$

$$f' = \frac{p'g - pg'}{g^2} = \frac{(2x+1)(x+1) - (x^2+x+2)\cdot 1}{(x+1)^2}$$

$$= \frac{2x^2+3x+1-x^2-x-2}{(x+1)^2} = \frac{x^2+2x-1}{(x+1)^2}$$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \qquad f = \frac{p}{g} \quad f' = \frac{p'g - pg'}{g^2}$$

$p(x) = \sin x \quad p'(x) = \cos x$

$g(x) = \cos x \quad g'(x) = -\sin x$

$$\frac{d}{dx}(\tan x) = \frac{\cos x \cos x - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$$

$$\boxed{\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}}$$

$$\frac{d}{dx}(\cot x) = \frac{d}{dt}\left(\frac{\cos x}{\sin x}\right) = \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$

$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} \Rightarrow$$

$$\boxed{\frac{d}{dx}(\cot x) = -\frac{1}{\sin^2 x}}$$

$$\frac{d}{dx}\left(\frac{f}{1}\right) = f' = \frac{f' \cdot 1 - f \cdot (0)}{1} = f' \checkmark$$

$$\frac{d}{dx}\left(\frac{1}{f}\right) = \frac{0 \cdot f - 1 \cdot f'}{f^2} = -\frac{f'}{f^2} \quad (*)$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2} \quad (f = x)$$

$$\frac{d}{dx}(x^n) = n x^{n-1} \qquad \frac{1}{x} = x^{-1}$$

$$\frac{d}{dx}(x^{-1}) = (-1)\, x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

Phew

$$\frac{d}{dx}(x^n) = n x^{n-1} \quad \text{for all } n \in \mathbb{Z}$$

integers

integers

$n \in \mathbb{N}$ $\qquad \dfrac{dx^n}{dx} = nx^{n-1}$ — derivative is always defined (at all x)

$n \in \mathbb{Z}$, $n<0$ $\qquad \dfrac{dx^n}{dx} = nx^{n-1} = n\dfrac{1}{x^{1-n}}$ $\qquad$ defined everywhere except $x=0$

Ex $\qquad \dfrac{d}{dx}\sec x = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right)$ $\qquad \dfrac{d}{dx}\left(\dfrac{1}{f}\right) = -\dfrac{f'}{f^2}$

$$\frac{d}{dx}\sec x = -\frac{(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos x}\frac{1}{\cos x} = \frac{\tan x}{\cos x} = \tan x \sec x$$

Ex $\qquad \dfrac{d}{dx}\csc x = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right) = \dfrac{0\cdot\sin x - 1\cdot\cos x}{\sin^2 x}$

$$\frac{d}{dx}\csc x = -\frac{\cos x}{\sin x}\frac{1}{\sin x} = -\cot x \csc x$$

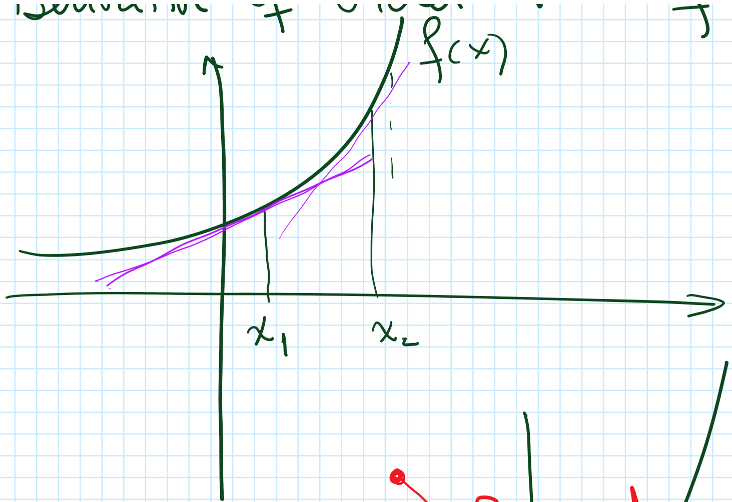
Higher derivatives $\qquad f:\mathbb{R}\to\mathbb{R}$

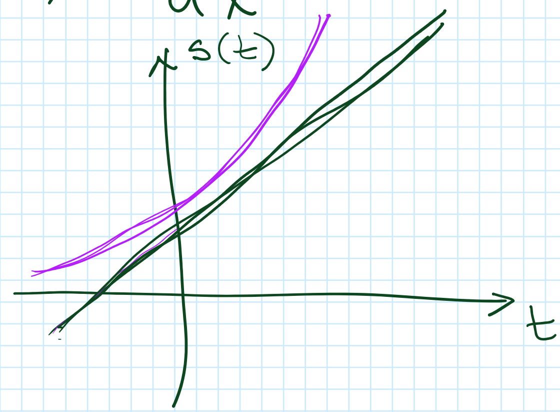
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad f = g'$$

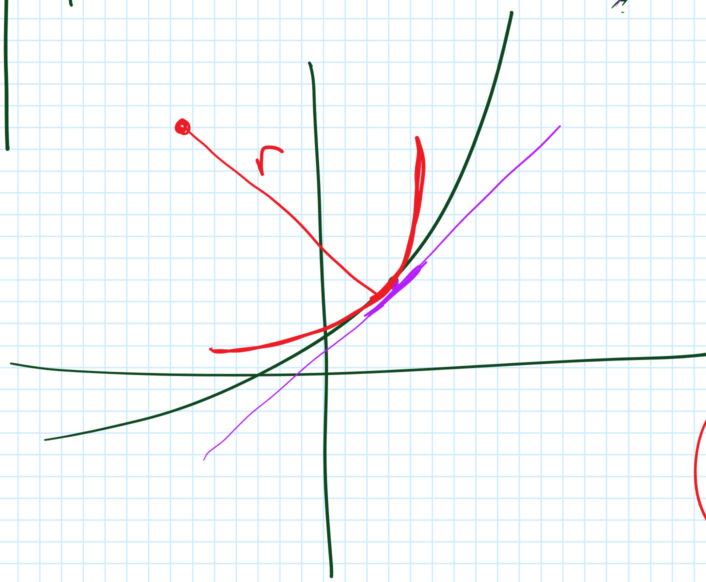
$$f'(x) = \frac{d}{dx}(g') = \underbrace{g'' = \frac{d^2g}{dx^2}}_{\text{Different notations}} = \lim_{h\to 0}\frac{g'(x+h)-g'(x)}{h}$$

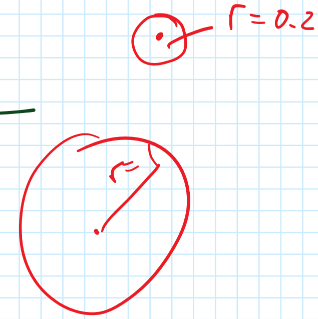
Derivative of order n $\qquad f^{(n)}(x) = \dfrac{d^n f}{dx^n}$.

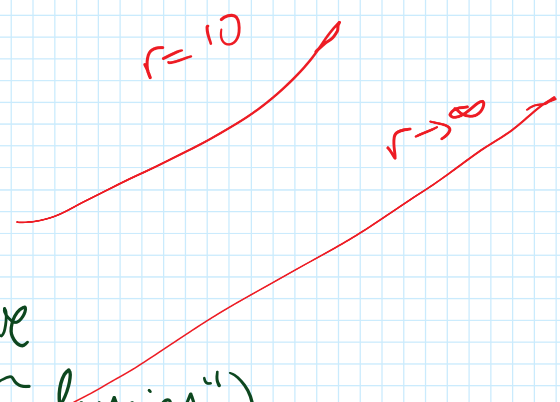
$f(x)$ $\qquad x\ s(t)$











Conclusion: Second derivative

→ acceleration ("physics")

→ curvature ("mathematics")

→ rate of change of a rate of change

Qualitative plot of a function considering the second derivative also

$f: \mathbb{R} \to \mathbb{R}$

$x \quad -\infty \quad x_4 \quad x_1 \quad x_2 \quad 0 \quad x_3 \quad x_5 \quad \infty$

x	$-\infty$	x_4	x_1	x_2	0	x_3	x_5	∞
$f(x)$	$-\infty$ $\searrow$	$\cdot$ $\nearrow$	0 $\nearrow$	0 $\nearrow$	$\nearrow$	0 $\nearrow$	$\nearrow$	∞
$f'(x)$	$-\infty$ $-$	0 $+$	$+$	$+$	$+$	$0+$ $+$ 0	$+$	∞
$f''(x)$								

II

Example of qualitative understanding function plots indicating a discontinuity.

III