More differentiation rules: Derivative of a composition of functions

$h(x) = f(g(x))$ h is the composition of f & g

$$h = f \circ g$$

Notice: Composition is different from multiplication

$h(x) = f(x) \cdot g(x)$ $h = f \cdot g$

$$h' = f'g + fg'$$

Simple example The Ferrari is going at 200 km/h
The McLaren is going twice as fast as the Ferrari

$x_F(t) = 200t$ $v_F = \frac{dx_F}{dt} = 200$

$$v_M(2u) = v_M(2v_F)$$

Rule for differentiation of composite functions

$h(x) = f(g(x))$ then if f', g' exist

$$h'(x) = f'(g(x))\, g'(x)$$

Procedure to evaluate

1) Determine outer/inner function f outer

g Inner

2) Denote $u = g(x)$

3) $h'(x) = f'(u)\, g'(x)$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Exercises

$h(x) = (5x+4)^3$

Procedure 1: $h(x) = 125x^3 + 3\cdot 25x^2 \cdot 4 + 3\cdot 5x \cdot 16 + 64$

$= 125x^3 + 300x^2 + 240x + 64$

$h'(x) = \underline{375x^2} + 600x + 240$ (lots of work!)

Procedure 2: Identify inner/outer

$h(x) = (5x+4)^3 = f(g(x))$

Inner $g(x) = 5x+4 = u$ $\qquad h(x) = f(u)$

$f(u) = u^3$

$h'(x) = f'(u)\, g'(x) = 3u^2\, 5 = 15(5x+4)^2$ (Much less work)

$= 15(25x^2 + 40x + 16) =$

$= 375x^2 + 600x + 240$

Exa. 2 $\qquad h(x) = \sqrt{5x^2+1} \qquad h(x) = f(g(x))$

$g(x) = 5x^2+1 = u \qquad g'(x) = 10x$

$f(u) = \sqrt{u} \qquad f'(u) = \dfrac{1}{2\sqrt{u}}$

$$f(u) = u^{\frac{1}{2}} \quad f'(u) = \frac{1}{2} u^{\frac{1}{2}-1} = \frac{1}{2\sqrt{u}}$$

$$h'(x) = f'(u)\, g'(x) = \frac{10x}{2\sqrt{5x^2+1}} = \frac{5x}{\sqrt{5x^2+1}}.$$

Exer 3 $\quad h(x) = \sin^3 x$

$$h(x) = f(g(x)) \qquad g(x) = \sin x = u \; ; \; g'(x) = \cos x$$

$$f(u) = u^3 \qquad f'(u) = 3u^2 = 3\sin^2 x$$

$$h'(x) = f'(u)\, g'(x) = 3\sin^2 x \cos x$$

Exer 4: $\quad h(x) = \sin x^3$

$$h(x) = f(g(x))$$

$$g(x) = x^3 = u \qquad g'(x) = 3x^2$$

$$f(u) = \sin u \qquad f'(u) = \cos u$$

$$h'(x) = (\cos x^3)\, 3x^2 = 3x^2 \cos x^3$$

Proof of rule $\quad h(x) = f(g(x)) \qquad u = g(x)$

$$h'(a) = \lim_{x \to a} \left(\frac{h(x) - h(a)}{x - a} \cdot 1 \right)$$

$$= \lim_{x \to a} \frac{h(x) - h(a)}{x - a} \cdot \left(\frac{g(x) - g(a)}{g(x) - g(a)} \right)$$

$$x \to a \quad \frac{\quad}{x-a} \quad g(x) - g(a)$$

$$= \lim_{x \to a} \frac{h(x) - h(a)}{g(x) - g(a)} \qquad \frac{g(x) - g(a)}{x - a}$$

$$= \qquad f'(u) \quad g'(x)$$