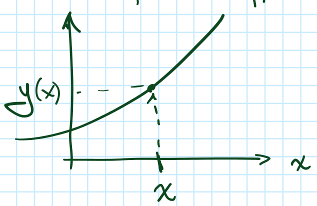
More differentiation rules: implicit differentiation

Tuesday, September 27, 2022 9:23 AM

$y : \mathbb{R} \to \mathbb{R}$ $\quad y(x)$

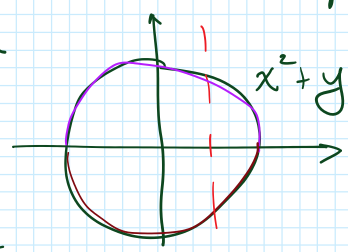


x is the independent variable

y is the dependent variable

explicitly in terms of y

Motivating example: circle



$x^2 + y^2 = 1$ "A relationship between x & y"

$y^2 = 1 - x^2 \qquad -1 \le x \le 1$

$|x| \le 1$

$$y(x) = \pm\sqrt{1-x^2} = \begin{cases} \sqrt{1-x^2} \\ -\sqrt{1-x^2} \end{cases}$$

First branch $\quad y(x) = \sqrt{1-x^2} = g(h(x)) \qquad g(u) = \sqrt{1-u}$

$h(x) = 1 - x^2$

$$y'(x) = g'(u)\,h'(x) = \frac{(-1)}{2\sqrt{1-u}}(-2x) \qquad g'(u) = \frac{(-1)}{2\sqrt{1-u}}$$

$$= \frac{\cancel{2}x}{\cancel{2}\sqrt{1-u}} = \frac{x}{\sqrt{1-x^2}} \qquad h'(x) = -2x$$

Error-prone (involves multiple applications of rules)

$x^2 + y^2 = 1 \qquad$ (Implicit relationship)

"Differentiate the relationship" $\Rightarrow$

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1) \ ; \quad \frac{d}{dx}(1) = 0$$

$$\frac{d}{dx}(x^2+y^2) = \frac{dx^2}{dx} + \frac{dy^2}{dx} = 2x + 2y\frac{dy}{dx}$$

$$\frac{dy^2}{dx} = \frac{d}{dx}a(b(x)) \qquad a(u) = u^2 \quad b(x) = y(x) \quad u = y(x)$$

$a'(u) = 2u \quad b'(x) = y'(x)$

$$\frac{dy^2}{dx} = a'(u)\,b'(x) = 2y(x)\,y'(x)$$

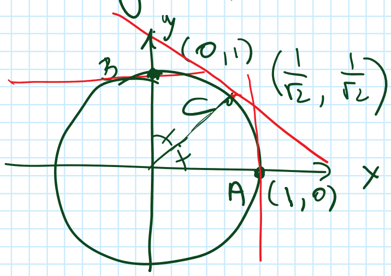
$$2x + 2yy' = 0 \Rightarrow x + yy' = 0$$

Find y' by algebra $\qquad y' = -\frac{x}{}$

Find y' by algebra $\qquad y' = -\dfrac{x}{y}$

$$y = \pm\sqrt{1-x^2}$$

Ex:



$(\sqrt{2}, \sqrt{2})$

$x^2 + y^2 = 4$

Tangent lines at A, B, C

Geometry inspection

at A: $\quad x = 1$

at B: $\quad y = 1$

at C: $\quad \left(y - \frac{1}{\sqrt{2}}\right) = (-1)\left(x - \frac{1}{\sqrt{2}}\right)$

$y' = -\dfrac{x}{y}$ $\quad$ Evaluate at A: $\quad y \to 0 \quad y' \to \infty$;

$\downarrow$

slope of tangent

B: $\quad y' = 0 \quad y(x) = \text{constant} = 1$

C: $\quad y' = -\dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = -1$

Ex: $\qquad \sin(xy) = x^2 + y^2$

Implicit differentiation of relation

$$\frac{d}{dx}\sin(xy) = \frac{d}{dx}(x^2 + y^2) = 2x + 2yy'$$

$$\frac{d}{dx}\sin(xy) = \cos(xy)\left(\frac{d}{dx}(xy)\right)$$

$$= \cos(xy)\left(\frac{dx}{dx}\cdot y + x\frac{dy}{dx}\right)$$

$$= \cos(xy)\left(y + xy'\right)$$

$\Rightarrow$

$$\cos(xy)(y + xy') = 2x + 2yy'$$

Do the algebra to find y'

$$y\cos(xy) + xy'\cos(xy) = 2x + 2yy'$$

$$xy'\cos(xy) - 2yy' = -y\cos(xy) + 2x$$

$$[x\cos(xy) - 2y]\, y' = 2x - y\cos(xy) \Rightarrow$$

$$y' = \frac{2x - y\cos(xy)}{x\cos(xy) - 2y}$$

Obs: Typical implicit diff. result y' is given in terms of both x and y

Ex: $x^2 + xy - y^2 = 7$

Differentiate $\frac{d}{dx}$: $2x + y + xy' - 2yy' = 0$

$$(x - 2y)y' = -2x - y$$

$$y' = \frac{2x + y}{2y - x}$$

Tangent at $(3,2)$

$$(y - y_1) = m(x - x_1)$$

$$(y - 2) = m(x - 3) \Rightarrow$$

$$m = y'(2) = \frac{2\cdot 3 + 2}{2\cdot 2 - 3} = \frac{8}{1} = 8$$

Tangent at $(3,2)$ is $y - 2 = 8(x - 3)$

Ex: Second derivative $x^2 + y^2 = 1$

Find y'' by implicit differentiation

$$(fg)' = f'g + fg'$$

Take $\frac{d}{dx}$: $2x + 2yy' = 0$

Take $\frac{d}{dx}$: $\cancel{2} + \cancel{2}(y'^2 + yy'') = 0$

$1 + y'^2 + yy'' = 0 \Rightarrow yy'' = -1 - y'^2$

$$yy'' = -1 - y'^2$$

$$y'' = -\frac{1 + y'^2}{y}$$

Have fun with implicit diff. in real-world problems & chain rule

T **104.** **A mixing tank** A 500-liter (L) tank is filled with pure water. At time $t = 0$, a salt solution begins flowing into the tank at a rate of 5 L/min. At the same time, the (fully mixed) solution flows out of the tank at a rate of 5.5 L/min. The mass of salt in grams in the tank at any time $t \ge 0$ is given by

$$M(t) = 250(1\,000 - t)(1 - 10^{-30}(1\,000 - t)^{10})$$

and the volume of solution in the tank is given by

$$V(t) = 500 - 0.5t.$$

a. Graph the mass function and verify that $M(0) = 0$.

b. Graph the volume function and verify that the tank is empty when $t = 1\,000$ min.

c. The concentration of the salt solution in the tank (in g/L) is given by $C(t) = M(t)/V(t)$. Graph the concentration function and comment on its properties. Specifically, what are $C(0)$ and $\lim_{t \to 1000^-} C(t)$?

d. Find the rate of change of the mass $M'(t)$, for $0 \le t \le 1\,000$.

e. Find the rate of change of the concentration $C'(t)$, for $0 \le t \le 1\,000$.

f. For what times is the concentration of the solution increasing? Decreasing?

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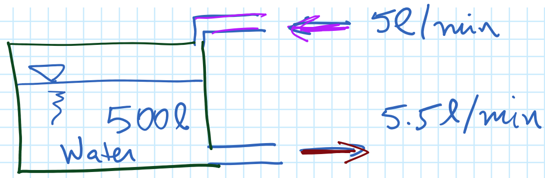


$$M(t) = 250\,(1000-t)\left[1 - 10^{-30}\,(1000-t)^{10}\right]$$

a)

t	0	1000
$M(t)$	0	
$M'(t)$		

$$M(0) = 250 \cdot 10^3\left[1 - 10^{-30} \cdot 1000^{10}\right]$$
$$= 250 \cdot 10^3\left[1 - 10^{-30}(10^3)^{10}\right]$$
$$= 250 \cdot 10^3\,[1-1] = 0$$

(Finish on Th.)

$$(fg)' = f'g + fg'$$

$$M'(t) = 250\left[(-1)\left[1 - 10^{-30}(1000-t)^{10}\right] + (1000-t)\left(-10^{-30}\,10\,(1000-t)^9(-1)\right)\right]$$
$$= 250\left[10^{-30}(1000-t)^{10} - 1 + 10^{-29}(1000-t)(1000-t)^9\right]$$
$$= 250\left[10^{-30}(1000-t)^{10} - 1 + 10^{-29}(1000-t)^{10}\right]$$
$$= 250\left[11 \times 10^{-30}(1000-t)^{10} - 1\right]$$

$$10^{-30} + 10^{-29} = 10^{-30} + 10 \cdot 10^{-30} = (1+10)\,10^{-30} = 11 \times 10^{-30}.$$

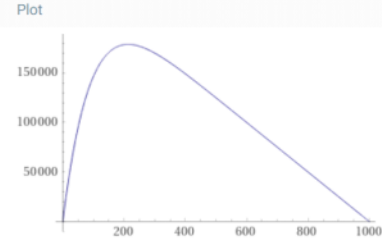
M(t)=250(1000-t)(1-10^(-30) (1000-t) ^10) for 0<=t<=1000

NATURAL LANGUAGE MATH INPUT EXTENDED KEYBOARD EX

Input interpretation

plot	$M(t) = 250\,(1000-t)\left(1 - \frac{(1000-t)^{10}}{10^{30}}\right)$	$t = 0$ to 1000

Plot



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Final technology example (good as a verification, does not replace the need to do this by hand)

Example: $y(x) = \sin x$ $\qquad \frac{dy}{dx} = \cos x$

Explicit $y(x) = \sin x$ $\qquad \frac{dy}{dx} = \cos x$

Implicit $F(x, y) = 0$

$y = \sin(xy)$ $\qquad y(x)$ $\qquad y: \mathbb{R} \to \mathbb{R}$

$\frac{d}{dx}:\ \left(y = \sin(xy) \right)$ $\qquad (xy)' = y + xy'$

$$y' = \cos(xy)(y + xy') = xy'\cos(xy) + y\cos(xy)$$

$y' - xy'\cos(xy) = y\cos(xy)$ "just as simple"

$[1 - x\cos(xy)]y' = y\cos(xy)$

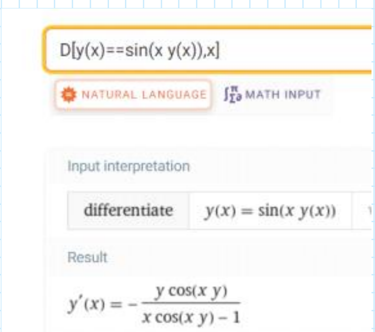
$$y' = \frac{y\cos(xy)}{1 - x\cos(xy)}$$

(valid $1 - x\cos(xy) \neq 0$)

Simple

Remember: convention
$a \cos x$ meaning $a \cdot \cos(x)$
$\cos x\, a$ misunderstood as $\cos(xa)$
$a \cos x$

$m + x = n + 2x \quad +(-n)$
$m - n + x = n + (-n) + 2x$
$m - n + x + (-x) = 0 + 2x + (-x)$
$m - n = x$



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