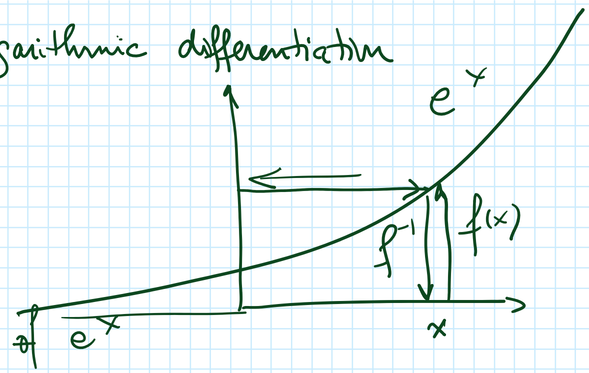
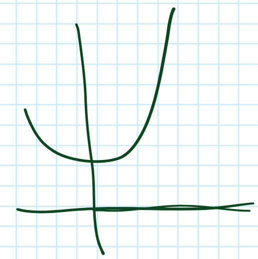
Thursday, September 29, 2022 9:50 AM

Derivative of log, logarithmic differentiation

$f(x) = e^x = y$

One-to-one function





f^{-1} inverse function of e^x

$y = \ln x$ inverse of exponential

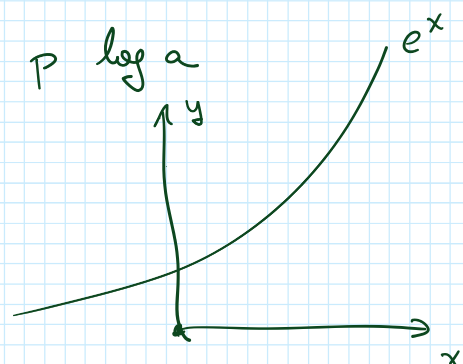
$x > 0$

(Precalc) Review of properties of logarithms

$\log(ab) = \log a + \log b$ (Slide rule)

$\log a^2 = \log(a \cdot a) = \log a + \log a = 2 \log a$

$\log a^p = p \log a$



log — logarithm in an arbitrary

ln — natural logarithm

lg — decimal log

$\ln y$

$\boxed{e^{\ln y} = y}$

Take ln on both sides

$\ln e^{\ln y} = \ln y$

$\ln y = \ln y$

$\ln e^{\ln y} = \ln e^a = a \ln e =$

$= \ln y \ln e = \ln y \, 1 = \ln y$

Derivative of $\ln x$

$y = \ln x$ $\quad x = e^y \quad \ln x = \ln e^y = y$

$\left(\frac{d}{dx} e^x = e^x \right)$

$x = e^y = \exp(y(x))$

$\frac{d}{dx}: \quad 1 = e^y y' = x y' \implies$

$\frac{d}{dx}$: $\quad 1 = e^{y} y' = x y' \Rightarrow$

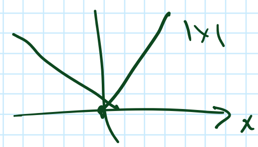
$$y' = \frac{1}{x}$$

$$\boxed{\frac{d}{dx} \ln x = \frac{1}{x} \quad x > 0}$$

$$y(x) = \ln|x| = \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases}$$

Precalc rev

$$|x| = \begin{cases} x & \\ & x < 0 \end{cases}$$



$$y'(x) = \begin{cases} \frac{1}{x} & x > 0 \\ \frac{1}{(-x)} \cdot (-1) = \frac{1}{x} & x < 0 \end{cases}$$

$x < 0$

$$\frac{d}{dx} \ln(-x) = \frac{1}{(-x)} \cdot (-1)$$

$$\frac{1}{(-x)} - 1 = -\frac{1}{x} - 1$$

$$\frac{d}{dx} \ln x = \frac{1}{x} \quad x > 0 \qquad \frac{d}{dx} \ln|x| = \frac{1}{x} \quad x \neq 0$$

$y(x) = a^x$

When $a = e$ $\quad y(x) = e^x \quad y'(x) = e^x$

$\left(2^x \; e^x \; 3^x \text{ reminder} \right)$

$a^x = y$ $\quad$ Recall $\quad e^{\ln u} = u$

Choose u

Choose $u = \underbrace{a^x}_{?}$

$$e^{\ln a^x} = a^x$$

$$e^{\ln y} = a^x = y$$

$$e^{\ln a^x} = e^{x \ln a} = y(x)$$

$$e^{x\ln a} = e^{x \ln a} = y(x)$$

$$\frac{dy(x)}{dx} = \frac{d}{dx} e^{x\ln a} = \frac{d}{dx}\exp(x\ln a) = e^{x\ln a}\frac{d}{dx}(x\ln a) =$$

$$= e^{x\ln a}\ln a = y\ln a$$

$$\boxed{\frac{d}{dx}(a^x) = \ln a\, a^x}$$

Check $a=e$

$$\frac{d}{dx}(e^x) = \ln e \cdot e^x$$

$e = 2.71$

Ex 1 $f(x) = 2^x = y$ $\frac{dy}{dx} = \ln 2 \; 2^x$

$\ln 2 < 1$

$\ln e = 1$

Ex 2 $f(x) = 3^x = y$ $\frac{dy}{dx} = \ln 3 \; 3^x$

$\ln 3 > 1$

Examples

Ex: Find derivative of $y(x) = \ln|\sin(x)|$

$x = -\pi$

$x \neq 0$

$x = \pi$

in general

$x \neq k\pi$

$k \in \mathbb{Z}$

$$y(x) = \ln|\sin(x)| = f(g(x))$$

$$f(u) = \ln|u| \qquad f'(u) = \frac{1}{u}$$

$$g(x) = \sin(x) \Rightarrow g'(x) = \cos x$$

$$y'(x) = \frac{\cos x}{\sin x} = \cot x$$

Ex:

$$y(x) = \ln(1+x^2)$$

$$y' = \frac{2x}{1+x^2}$$

Derivative of $y(x) = x^p$ for general p

$$y'(x) = \lim_{h\to 0} \frac{y(x+h)-y(x)}{h} = \lim_{h\to 0} \frac{(x+h)^p - x^p}{h}$$

$$a^2-b^2 = (a-b)(a+b)$$

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

$p \in \mathbb{N}$ $$a^p - b^p = (a-b)(a^{p-1}b^0 + a^{p-2}b + \dots + a^0 b^{p-1})$$

Review L05 or L06 $$y'(x) = \lim_{h\to 0} \frac{\cancel{h}(x \qquad)}{\cancel{h}} = p x^{p-1}$$

$$y(x) = x^p \qquad p \in \mathbb{R} \qquad x > 0$$

$$e^{\ln u} = u \qquad u = x^p \qquad x^p = e^{\ln x^p} = e^{p \ln x}$$

$$y(x) = x^p = e^{p\ln x}$$

$$y'(x) = \frac{d}{dx}\left[\exp(p \ln x)\right] = e^{p \ln x} \frac{d}{dx}(p \ln x)$$

$$\Rightarrow \quad \boxed{y'(x) = e^{p\ln x}\frac{p}{x} = x^p \frac{p}{x} = p x^{p-1}}$$

for $n \in \mathbb{N}$ $\quad \frac{d}{dx} x^n = n x^{n-1}$ (Recall long proof based on def. of deriv & alg. identity $a^n - b^n = (a-b)(a^{n-1} + \dots$

Ex:

$f(x) = \underbrace{x^e}$	$g(x) = \underline{e^x}$	$h(x) = \underbrace{\pi^x}$	$u(x) = x^\pi$
$\frac{df}{dx} = e x^{e-1}$	$\frac{dg}{dx} = \ln e \, e^x$	$\frac{dh}{dx} = \ln\pi \, \pi^x$	$u' = \pi x^{\pi - 1}$
$x > 0$	$\frac{dg}{dx} = 1 \cdot e^x$		$x > 0$

$x > 0$ $\frac{dy}{dx} = 1 \cdot e^x$ $x > 0$

$$\frac{dy}{dx} = e^x$$

Logarithmic differentiation

$$f = f(x) = \frac{(1-2x^3)^3\sqrt{1+x}}{(1+x^2)^3} = \frac{A \cdot B}{C} \qquad f(x) = \frac{P(x)}{Q(x)}$$

$$f' \qquad\qquad f' = \frac{P'Q - PQ'}{Q^2}$$

Assume positive l.h.s & r.h.s.

Take ln of both sides

$$(*) \quad \ln f = 3\ln(1-2x^3) + \frac{1}{2}\ln(1+x) - 3\ln(1+x^2)$$

$$A = (1-2x^3)^3 \quad B = \sqrt{1+x} = (1+x)^{1/2} \quad C = (1+x^2)^3$$

Take $\frac{d}{dx}$ of both sides of $(*)$

$$\frac{f'}{f} = 3\,\frac{(-6x^2)}{1-2x^3} + \frac{1}{2}\,\frac{1}{1+x} - 3\,\frac{2x}{1+x^2}$$

$$f' = f \cdot \left(\qquad \text{——} \, " \, \text{——} \qquad \right)$$

Ex: Recover quotient rule

$$y(x) = \frac{P(x)}{Q(x)} \qquad y'(x) = \frac{P'(x)Q(x) - P(x)Q'(x)}{Q^2(x)}$$

Apply log diff.

$$\ln y = \ln P - \ln Q$$

$$\frac{y'}{y} = \frac{p'}{p} - \frac{q'}{q} = \frac{p'q - pq'}{pq}$$

$$y' = y\,\frac{p'q - pq'}{pq} = \frac{\cancel{p}}{q}\,\frac{p'q - pq'}{\cancel{p}q} = \frac{p'q - pq'}{q^2}.$$

Yay! Math is self-consistent.

Ex: $f(x) = \dfrac{x^8 \cos^3 x}{\sqrt{x-1}}$ (in general $\dfrac{x^n \cos^p x}{\sqrt{x^2-1}}$ planet orbit calculation) Precalc review

Want to compute $f'(x)$

Expedient to take log (natural)

$$\ln f = \ln \frac{x^8 \cos^3 x}{\sqrt{x-1}}$$

- $\ln A^p = p \ln A$
- $\ln AB = \ln A + \ln B$
- $\ln \frac{AB}{C} = \ln A + \ln B - \ln C$
- $\ln \frac{1}{C} = \ln C^{-1} = (-1) \ln C$

$$\ln f = \ln x^8 + \ln \cos^3 x - \ln \sqrt{x-1}$$

$$\frac{d}{dx}: \ln f = 8 \ln x + 3 \ln(\cos x) - \frac{1}{2} \ln(x-1)$$

(Assume $x > 1$)

$\ln(\cos x) = c(x)$

$c(x) = a(b(x))$

$b(x) = \cos x$

$a(u) = \ln u$

$c'(x) = a'(u)\, b'(x)$

$$\frac{d}{dx} \ln f = \frac{d}{dx} 8 \ln x + \frac{d}{dx} 3 \ln(\cos x) - \frac{1}{2} \frac{d}{dx} \ln(x-1)$$

$$\frac{d}{dx} \ln f = 8 \frac{1}{x} + 3 \frac{1}{\cos x}(-\sin x) - \frac{1}{2} \frac{1}{x-1}$$

$$\frac{f'}{f} = \frac{8}{x} - 3 \tan x - \frac{1}{2} \frac{1}{x-1}$$

$$f'(x) = \frac{x^8 \cos^3 x}{\sqrt{x-1}} \left[\frac{8}{x} - 3 \tan x - \frac{1}{2} \frac{1}{x-1} \right].$$