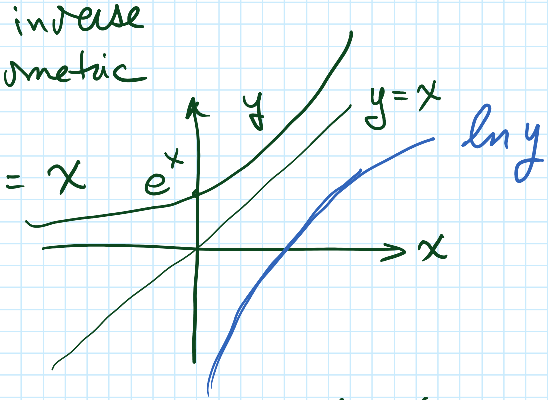
Derivatives of inverse functions ; of inverse trigonometric

Tuesday, October 4, 2022 9:49 AM

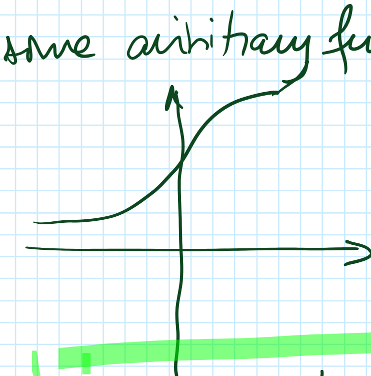
Inverse function : $y = e^x$ $\ln y = x$



$y(x) = x$ $y = x$

Generalise to derivative of inverse of some arbitrary function

$y = f(x)$ $x = f^{-1}(y)$



f^{-1} exists only if f is one-to-one

Compute derivative of inverse f^{-1}

$$f^{-1}(y) \neq \frac{1}{f(y)}$$

$$[f(y)]^{-1} = \frac{1}{f(y)}$$

$$\frac{d}{dy}\left[f^{-1}(y_0)\right] = \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0}$$

$x = f^{-1}(y)$; $x_0 = f^{-1}(y_0)$

$y = f(x)$; $y_0 = f(x_0)$ $\Rightarrow$

$y \to y_0 \Rightarrow x \to x_0$

assume f differentiable, continuous

$$\frac{d}{dy}\left[f^{-1}(y_0)\right] = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)}$$

$$\frac{d}{dy}\left[f^{-1}(y_0)\right] = \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}$$

$$\frac{d}{dy}\left[f^{-1}(y_0)\right] = \frac{1}{f'(x_0)}$$

$$\frac{d}{dy}\left[f^{-1}(y_0)\right] = \frac{1}{f'(x_0)}$$

In general, for some arbitrary (x, y)

$$\frac{df^{-1}}{dy} = \frac{1}{\frac{df}{dx}}$$

"Derivative of inverse function = reciprocal of derivative of function"

"Derivative of inverse function = inverse of function derivative"

inverse of function derivative $= \frac{1}{f'}$.

Recall $\quad f'(x_0) = \lim \dfrac{f(x) - f(x_0)}{x - x_0}$

$$a'(t_0) = \lim \frac{a(t) - a(t_0)}{t - t_0} : \left\{ \begin{array}{l} \text{Possibilities} \\ \cancel{\text{"}\infty\text{"}} \\ \cancel{\text{"}-\infty\text{"}} \\ \text{~~"undetermined"~~} \end{array} \right.$$

$$\lim_{\theta \to \theta_0} e^{\frac{\sin\theta - \sin\theta_0}{\theta - \theta_0}} = e^{\sin'(\theta_0)} = e^{\cos\theta_0}$$

$$\lim_{\theta \to \theta_0} \frac{\sin\theta - \sin\theta_0}{\theta - \theta_0} = \sin'\theta_0 = \cos\theta_0$$

$$\lim \boxed{\left(\frac{x - x_0}{\ }\right)}^{\frac{t^2 - t_0^2}{t - t_0}} = \lim \left(\frac{1}{\ }\right)^{\frac{(t - t_0)(t + t_0)}{t - t_0}} =$$

$$\lim_{t\to t_0} \boxed{\left(\frac{x-x_0}{x^2-x_0^2}\right)} = \lim_{t\to t_0}\left(\frac{1}{x+x_0}\right) =$$

$$= \lim_{t\to t_0}\left(\frac{1}{x+x_0}\right)^{(t+t_0)} = \frac{1}{(x+x_0)}2t_0 \checkmark$$

$$\lim_{t\to t_0}\frac{t^2-t_0^2}{t-t_0} = \left(\frac{d}{dt}t^2\right)(t_0) = 2t_0 \checkmark$$

In general $y = f(x)$ f invertible & has derivative

$x = f^{-1}(y)$

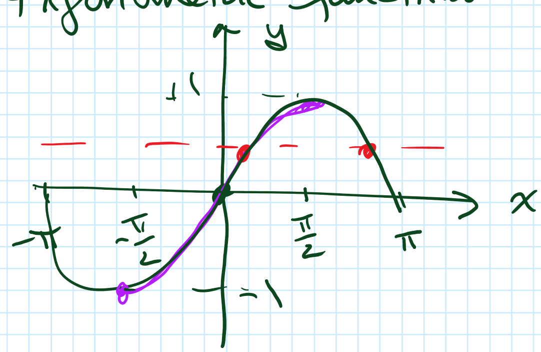
$$\frac{df^{-1}}{dy} = \frac{1}{\frac{df}{dx}}$$

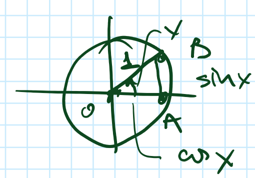
— Derivatives of inverse trigonometric functions

$y = \sin x$

$f(x) = \sin x$

$f: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$





Remember: "a function = domain, co-domain, procedure to associate values"

f is one-to-one $\Rightarrow$ f^{-1} exists

$f^{-1}: [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ $\boxed{\sin^{-1} y = x}$

$f^{-1}(y) = x$;

derivative of $f^{-1}(y) = \frac{d}{dy} f^{-1}(y) = \frac{dx}{dy}$

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$\sin^{-1} y = x \Rightarrow \sin x = y$ Implicit relation

$\frac{d}{dy}$: $\sin x = y$

$f^{-1}(y) = x$

$$\frac{d}{dy} \sin x = \frac{dy}{dy}$$

$$(\cos x) \boxed{\frac{dx}{dy}} = 1$$

$$\boxed{\frac{d}{dy} \sin(x(y)) = (\cos x) \frac{dx}{dy}}$$

This is the derivative $\frac{d}{dy} \sin^{-1} y$

$$\frac{dx}{dy} = \frac{1}{\cos x}$$

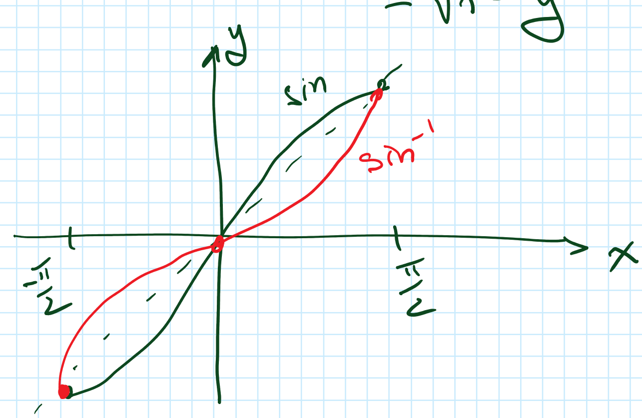
Trigonometric identity

$\cos^2 x + \sin^2 x = 1 \Rightarrow$

$\cos^2 x + y^2 = 1 \Rightarrow$

$\cos x = \pm\sqrt{1 - y^2}$

$$\frac{dx}{dy} = \frac{1}{\sqrt{1-y^2}}$$



$$\frac{d}{dy} \sin^{-1} y = \frac{1}{\sqrt{1-y^2}}$$

Admitedly: there are many more things that are more fun.

Differentiation tables

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$