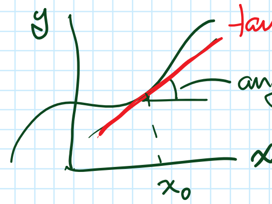
Thursday, October 6, 2022 9:31 AM

Related derivatives

$f : \mathbb{R} \to \mathbb{R}$ $\qquad y = f(x)$



angle θ ; $\tan\theta =$ slope $= f'(x_0)$

Related derivatives $\quad f, g : \mathbb{R} \to \mathbb{R}$

$\underbrace{\text{Relate } f' \text{ to } g'}_{\text{natural language}}$. $\quad F(f', g') = 0 \quad$ Mathematical statement

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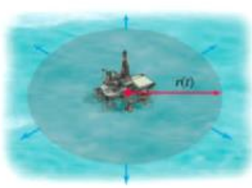
EXAMPLE 1 Spreading oil

An oil rig springs a leak in calm seas and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of 30 m/hr, how fast is the area of the patch increasing when the patch has a radius of 100 m (**Figure 3.77**)?



$t =$ 3 hr

radius = 90 m

$\frac{dr}{dt} =$ 30 m/hr

area = $8\,100\,\pi$ m^2

$\frac{dA}{dt} =$ $5\,400\,\pi$ m^2/hr

t 1 2 3 4 5

$\frac{dr}{dt}$ 25 30 35

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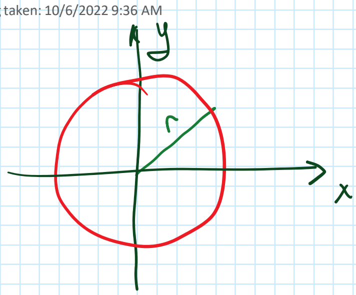
(Looking from above)



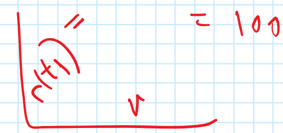
r : radius (dep var)

t : time (ind var.)

Rate $\to$ derivative $\quad r'(t) = 30 \frac{m}{h}$

Area: $\quad \boxed{A(r) = \pi r^2}$

Artificial



at t_1: $\quad r(t_1) = 100 \qquad$ Find t_1

Need to find $r(t)$. $\quad r'(t) = 30$

Pick origin for time $\quad r(0) = 0$

$\Rightarrow \quad r(t) = 30t$

$r(t_1) = 100 \Rightarrow 30t_1 = 100$

"Implies"

Differentiate

$f(t) = 30t + 100$

$g(t) = 30t - 2\times10^9$

$f(0) = 100$

$t_1 = \frac{10}{3}$ h

Rate of increase of area: $\frac{d}{dt} A(r(t))$ $\quad$ composite func. diff

$\frac{d}{dt} A(r(t)) = A'(r(t))\, r'(t)$, $\quad \Rightarrow$

dA ... dt

$$\frac{d}{dt} A(r(t)) = A'(r(t))\, r'(t)$$

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r; \quad v = \frac{dr}{dt} = 30$$ $\Rightarrow$

$$\frac{dA}{dt} = 2\pi r \cdot 30 = 60\pi r =$$

(Take units $\frac{dA}{dt} = 2\pi r\, v \qquad \frac{m^2}{h} = \frac{m\ m}{h}$ ✓)

$\frac{dA}{dt}$ when $r = 100$ $\therefore$ $\frac{dA}{dt} = 6000\pi$.

$\frac{dA}{dt} = 6000\pi$ (with no justification) $\Rightarrow$ 0 pts

! provide motivation as above

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32. Drinking a soda At what rate (in in^3/s) is soda being sucked out of a cylindrical glass that is 6 in tall and has a radius of 2 in? The depth of the soda decreases at a constant rate of 0.25 in/s.

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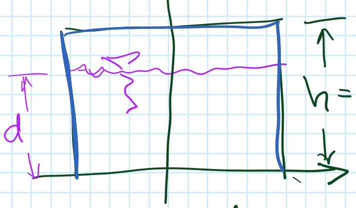
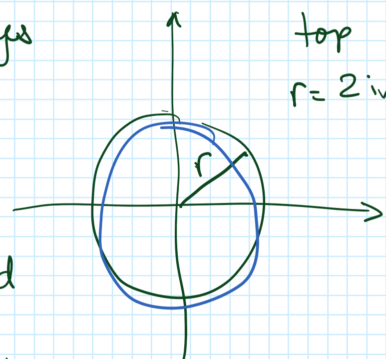




volume : V soda depth : d

time : t

$V(t)$ $d(t)$

Find $V'(t)$. Know $d'(t) = v = -0.25\ \frac{in}{s}$

Decreasing $d'(t) < 0$

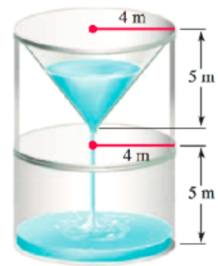
$V = \pi r^2 d$ volume of soda

$V(t) = \pi r^2 d(t)$

$$V'(t) = \frac{dV}{dt} = \pi r^2 d'(t) = \pi r^2 v$$

$$V'(t) = \pi \cdot 4 \cdot \left(-\frac{1}{4}\right) = -\pi \ \frac{in^3}{s}.$$ ✓

38. Two tanks A conical tank with an upper radius of 4 *m* and a height of 5 *m* drains into a cylindrical tank with a radius of 4 *m* and a height of 5 *m* (see figure). If the water level in the conical tank drops at a rate of 0.5 *m*/min, at what rate does the water level in the cylindrical tank rise when the water level in the conical tank is 3 *m*? 1 *m*?



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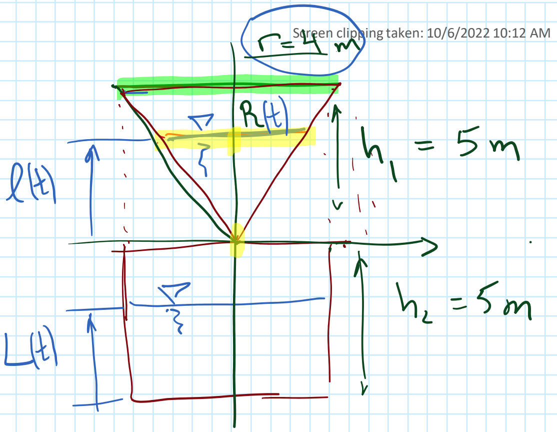


ℓ : level in conical tank

$\ell'(t) = v = -0.5 \frac{m}{min}$

L : level in cylindrical tank

$R(t)$: radius of base of cone of water

$V(t)$: volume of water in conical tank

$W(t)$: volume of water in cylindrical tank

change in volume (Water from conical tank) = change in volume (cylindrical)

$\ominus$ $\oplus$

$$-\frac{dV}{dt} = \frac{dW}{dt} \quad (\text{"Conservation of mass"})$$

$$\cancel{V(t) = \frac{1}{3}\pi r^2 \ell(t)}$$

$$V(t) = \frac{\pi}{3} R^2(t)\, \ell(t) \qquad W(t) = \pi r^2 L(t)$$

Find: Rate at which water level rises in cyl. tank $\cancel{\left(\frac{dW}{dt}\right)}$

$$W' = \pi r^2 L'(t)$$

$$\frac{dL}{dt} = \frac{1}{\pi r^2} W' = -\frac{1}{\pi r^2} V' \quad \Rightarrow$$

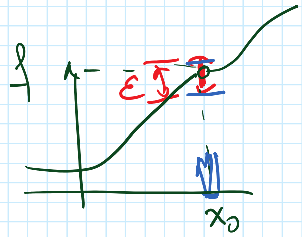
$$\Rightarrow \quad V' = \frac{\pi}{3}\left[2RR'\ell + R^2\ell'\right]$$

$$L' = -\frac{1}{\pi r^2}\frac{\pi}{3} R\left[2R'\ell + R\ell'\right] \quad \Rightarrow L' = \dots$$

$\ell' = -0.5 \quad r = 4$

$l' = -0.5 \quad r = 4m$

Chapter 3 Review



Chap.2 : Limit $\quad \lim_{x \to x_0} f(x) = f(x_0)$

$$\forall \varepsilon > 0 \quad \exists\, \delta_\varepsilon \quad \text{s.t.} \quad |x - x_0| < \delta_\varepsilon \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

Derivative at a point $\quad f: D \to C$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

Derivative function $\quad f'(x) = \frac{df}{dx}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\frac{d}{dx} x^n = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{\cancel{h}\left[(x+h)^{n-1} + (x+h)^{n-2} x + \dots + x^{n-1}\right]}{\cancel{h}}$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2} b + \dots + b^{n-1})$$

$$a = x+h \quad b = x \qquad a - b = h$$

$$\Rightarrow \frac{d}{dx} x^n = n\, x^{n-1}$$

Diff Table

f	f'
x^n	$n\, x^{n-1}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
e^x	e^x
a^x	$\ln a \, a^x$
$\ln x$	$\frac{1}{x}$

Diff rules

$$(f+g)' = f' + g'$$

$$(fg)' = f'g + fg'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$(f(g(x)))' = f'(g(x)) \cdot g'(x) = f'(u)\, g'(x)$$

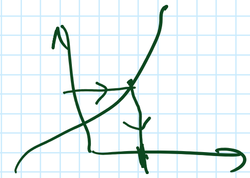
convenient $\quad u = g(x)$

$\ln x \quad \Big| \quad \frac{1}{x}$

convenient $u = g(x)$

f is one-to-one $\Rightarrow$

$\exists f^{-1}$

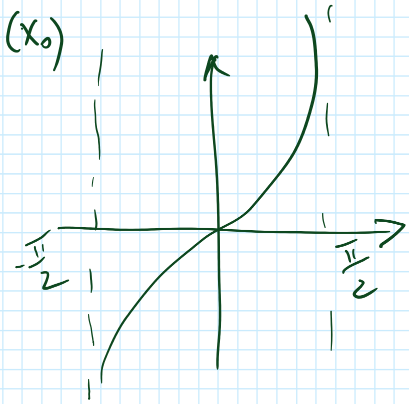


$$[f^{-1}(y_0)]' = \frac{1}{f'(x_0)}$$

Ex. (deriv. of inverse)

$$\frac{d}{dx}\arctan x = \frac{d}{dx}\tan^{-1} x$$

$y = \tan^{-1} x \qquad y : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$



$\Rightarrow \tan y = x$

$$\frac{d}{dx}\tan^{-1} x \Big| = \frac{1}{\frac{d}{dy}\tan y} = \frac{1}{\sec^2 y}$$

$$= \frac{1}{1+x^2}$$

$\tan^2 y + 1 = \frac{\sin^2 y}{\cos^2 y} + 1 =$

$\tan^2 y + 1 = \frac{\sin^2 y + \cos^2 y}{\cos^2 y}$

$\tan^2 y + 1 = \sec^2 y$

$x^2 + 1 = \sec^2 y$

Do the above "a bunch of times"