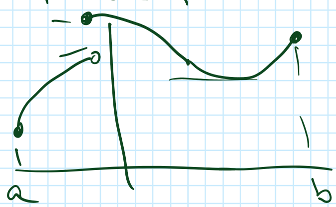
More examples on critical points & Rolle theorem
Mean Value Theorem (MVT), Information from derivative

Thursday, October 13, 2022 9:29 AM

Critical point examples

$f : [a,b] \to \mathbb{R}$ Critical points $\begin{cases} \text{where } f'(c)=0 \text{ (perhaps more than 1 pt)} \\ \text{where } f \text{ is not continuous} \end{cases}$



74. Maximizing revenue A sales analyst determines that the revenue from sales of fruit smoothies is given by $R(x) = -60x^2 + 300x$, where x is the price in dollars charged per item, for $0 \le x \le 5$.

a. Find the critical points of the revenue function.

b. Determine the absolute maximum value of the revenue function and give the price that maximizes the revenue.

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$R(x) = -60x^2 + 300x$ $\quad$ x = price per item

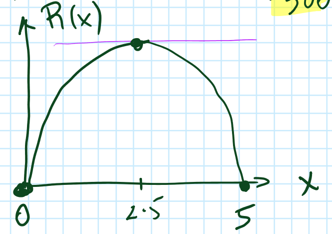
$R : [0,5] \to \mathbb{R}$ $\quad$ $R(x)$ = revenue

a. Critical points?

x	0	2.5	5	(ind. var.)
$R(x)$	0	$R(2.5)$	0	(dep. var.)
$R'(x)$	300 + + +	0	− − − −300	(rate of change)

$R'(x) = -120x + 300$

$R'(x_1) = 0$



↪ (Use a different notation to denote not the variable x, but the specific value when $R'=0$)

$-120x_1 + 300 = 0 \Rightarrow x_1 = \frac{300}{120} = 2.5$

b). price that maximizes revenue $= x_1 = 2.5$ \$/item

Revenue $= R(2.5)$.

76. Minimizing rectangle perimeters All rectangles with an area of 64 have a perimeter given by $P(x) = 2x + \frac{128}{x}$, where x is the length of one side of the rectangle. Find the absolute minimum value of the perimeter function on the interval $(0, \infty)$. What are the dimensions of the rectangle with minimum perimeter?

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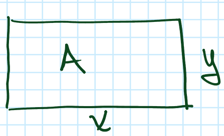


$A = 64$

$P(x) = 2x + \frac{128}{x}$ $\quad$ x length of a side

$P : (0,\infty) \to \mathbb{R}$

Minimum of P? Find x, y

$A(x) = xy = 64$

x	0	8	∞
$P(x)$	−		
$P'(x)$	− − −	0	+ + + +

$P'(x) = 2 - \frac{128}{x^2}$

x_1 $\quad$ $P'(x_1) = 0 \Rightarrow$

$2 - \frac{128}{x_1^2} = 0 \Rightarrow x_1^2 = 64$

$\Rightarrow x_1 = 8$

$\lim_{x \to 0} P(x)$ = undefined

$\lim_{x \to 0^+} P(x) = \infty$ $\qquad$ $\lim_{x \to 0^+} P'(x) = -\infty$

$2 - \frac{128}{x_1^2} = 0 \Rightarrow x_1^2 = 64$

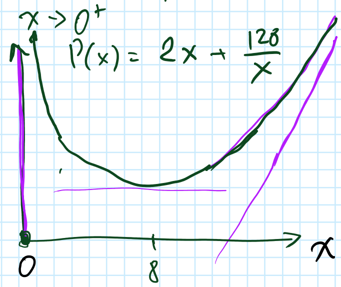
$\Rightarrow \quad x_1 = 8$

$\lim\limits_{x \to 0^+} P(x) = \infty \qquad \lim\limits_{x \to 0^+} P'(x) = -\infty$

$\lim\limits_{x \to \infty} P'(x) = 2$

$P(x) = 2x + \frac{128}{x}$

$= (\text{line}) + (\text{inversely proportional})$



For large x (inversely proportional) is small at large $P(x) \cong 2x$

$P(8) = 16 + \frac{128}{8} > 0 \qquad$ minimum value $(P'(8) = 0)$

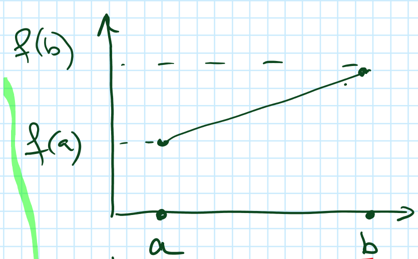
$\left.\begin{matrix} x = 8 \\ A = 64 \end{matrix}\right\} \Rightarrow \quad y = 8$

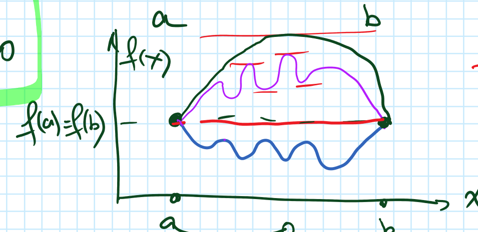
Rolle theorem

$\left.\begin{matrix} f : [a,b] \to \mathbb{R} \\ f(a) = f(b) \\ f \text{ differentiable} \end{matrix}\right| \Rightarrow \exists c \quad a < c < b \quad \text{s.t.} \quad f'(c) = 0$

$f' : (a,b) \to \mathbb{R}$

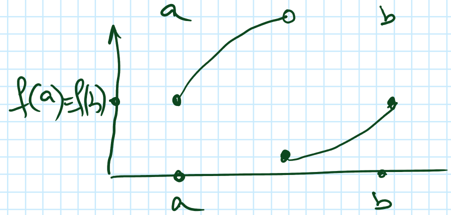
which one: $\cancel{[\quad]}$ include endpoints; $(\quad)$ ✓ exclude endpoints





$f(x) = \text{constant}$

$f'(x) = 0$



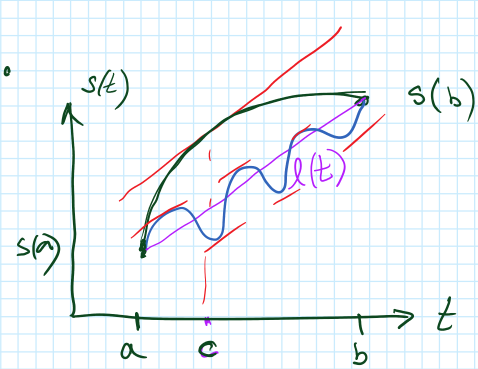
Rolle's Theorem

For $f : [a,b] \to \mathbb{R}$, $f' : (a,b) \to \mathbb{R}$, f differentiable with $f(a) = f(b)$ there must exist some c, $a < c < b$ such that

$$\boxed{f'(c) = 0}.$$

Mean Value Theorem

$\left.\begin{matrix} s : [a,b] \to \mathbb{R} \text{ continuous} \\ s \text{ differentiable} \\ s' : (a,b) \to \mathbb{R} \end{matrix}\right] \Rightarrow$



$\exists\, a < c < b$ s.t.

Average velocity $= \frac{s(b) - s(a)}{\quad} =$

$s' : (a,b) \to \mathbb{R}$ a c b

$\exists\, a < c < b$ s.t.

$$s'(c) = \frac{s(b) - s(a)}{b - a}.$$

Average velocity $= \dfrac{s(b) - s(a)}{b - a} =$ slope of

Proof. $g(t) = s(t) - \ell(t)$ $g(a) = 0$ $g(b) = 0$

Apply Rolle's theorem to $g(t) \Rightarrow$ $g'(t) = s'(t) - \ell'(t) \Rightarrow$

$\exists c$ $s'(c) = \ell'(c) = \dfrac{s(b) - s(a)}{b - a}$

Examples

11. $f(x) = x(x-1)^2$; $[0, 1]$

12. $f(x) = \sin 2x$; $[0, \pi/2]$

13. $f(x) = \cos 4x$; $[\pi/8, 3\pi/8]$

14. $f(x) = 1 - |x|$; $[-1, 1]$

15. $f(x) = 1 - x^{2/3}$; $[-1, 1]$

16. $f(x) = x^3 - 2x^2 - 8x$; $[-2, 4]$

17. $g(x) = x^3 - x^2 - 5x - 3$; $[-1, 3]$

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11. $f(x) = x(x-1)^2$ $f : [0,1] \to \mathbb{R}$

Q: Does R.T. apply?

f is a polynomial, hence continuous & differentiable

$f(0) = 0$ $f(1) = 0$ $\Rightarrow$ R.T. applies

$f'(c) = 0 \Rightarrow$

$$f'(x) = \frac{d}{dx}\left[x(x-1)^2\right] = 1 \cdot (x-1)^2 + x \cdot 2(x-1)$$

$[(fg)' = f'g + fg']$

To evaluate g' we applied chain rule

$f'(c) = 0 \Rightarrow$ $f'(c) = (c-1)^2 + 2c(c-1) = 0$

$$c^2 - 2c + 1 + 2c^2 - 2c = 0$$

$$3c^2 - 4c + 1 = 0 \qquad (3c-1)(c-1) = 0$$

$c_1 = \frac{1}{3}$; $c_2 = 1$

$0 < c < 1$

$\Rightarrow$ only solution is $c = \frac{1}{3}$.