Optimization

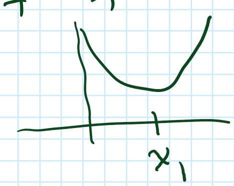
Technique:

1) Transform problem into a $f(x)$
(univariate) (differentiable)

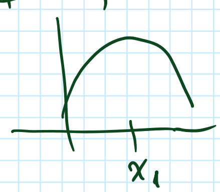
2) Look for local extrema of $f(x)$

$f'(x) = 0$ Equation $\Rightarrow$
say x_1 is a solution

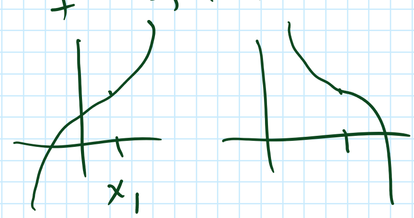
3) if $f''(x_1) > 0 \Rightarrow x_1$ is a local minimum



if $f''(x_1) < 0 \Rightarrow x_1$ is a local maximum

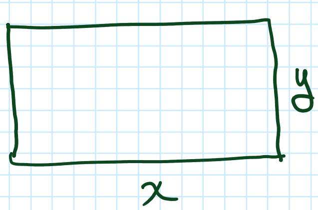


if $f''(x_1) = 0 \Rightarrow x_1$ is an inflection point



4) Check against end-points of the function domain

Ex. 1:



For given, fixed perimeter what is the maximum enclosed area?

Stated math. formulation

$A(x,y) = xy$ (Bivariate function)

$P(x,y) = 2(x+y)$ P_0: notation for fixed perimeter

Constraint: $P(x,y) = P_0 \Rightarrow 2(x+y) = P_0$

Eliminate a variable, y: $y = \frac{P_0}{2} - x$

$$A\Big|_{\text{constrained}}(x,y) = x\left(\frac{P_0}{2} - x\right) = S(x)$$

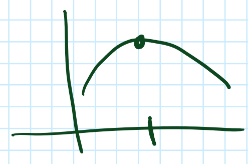
Step 1: $S(x) = x\left(\frac{P_0}{2} - x\right)$ Polynomial $\Rightarrow$ differentiable

Step 2: • $S'(x) = \frac{P_0}{2} - 2x = 0 \Rightarrow$

$x_1 = \frac{P_0}{4}$ is solution, possible local extremum

Step 3: $S''(x) = -2$

x_1 is a local maximum



Step 4: Check endpoints $S(x) = x\left(\frac{P_0}{2} - x\right)$ (Enclosed area)

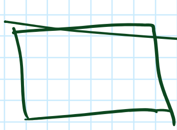
S Domain $S: \left[0, \frac{P_0}{2}\right] \to [0, S_{max}]$

$S(0) = 0$ $S\left(\frac{P_0}{2}\right) = 0 \Rightarrow$

Local maximum is also global maximum.

Maximum occurs at $x_1 = \frac{P_0}{4}$

Maximum is $S\left(\frac{P_0}{4}\right) = \frac{P_0}{4}\left(\frac{P_0}{2} - \frac{P_0}{4}\right) = \underline{\frac{P_0^2}{16}}$

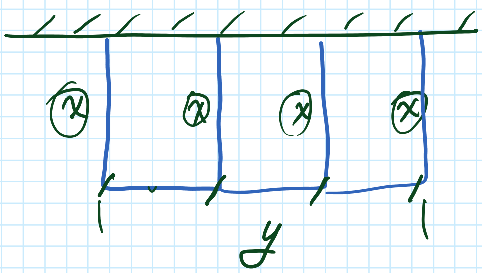
Suppose  ~~$P = 20$ $S(5) = \frac{400}{16}$~~

ALWAYS: Always use notation, and in last step introduce values.

$R = 25$ $\qquad\qquad$ $V = 2\pi R^2$

$V = 2\pi\, 625$

Ex 2: Three enclosures of maximum total area, knowing that one side is already enclosed, for fixed perimeter



Stage 1:

Vert. dim: x $\qquad$ Horiz dim: y

$$A(x,y) = xy \quad \left(\text{total enclosed area}\right)$$

$$P(x,y) = 4x + y \quad (\text{Perimeter})$$

P_0 (fixed perimeter), Constraint. $P(x,y) = P_0$

$$4x + y = P_0$$

Use constraint to eliminate y; $y = P_0 - 4x \Rightarrow$

$$A\Big|_{\text{constrained}}(x,y) = x(P_0 - 4x) = S(x) \quad \text{polynomial differentiable}$$

2) $S'(x) = P_0 - 8x \Rightarrow S'(x) = 0 \Rightarrow x_1 = \frac{P_0}{8}$

3) $S''(x) = -8 < 0 \Rightarrow x_1$ local maximum

$S(0) = 0 \qquad S\left(\frac{P_0}{4}\right) = 0 \Rightarrow x_1$ global maximum

Maximum occurs at $x_1 = \frac{P_0}{8}$

Maximum is $S(x_1) = \frac{P_0}{8}\left(P_0 - \frac{P_0}{2}\right) = \frac{P_0^2}{16}$.

Ex 3: $\qquad$ B $\qquad$ Start from A to arrive at B

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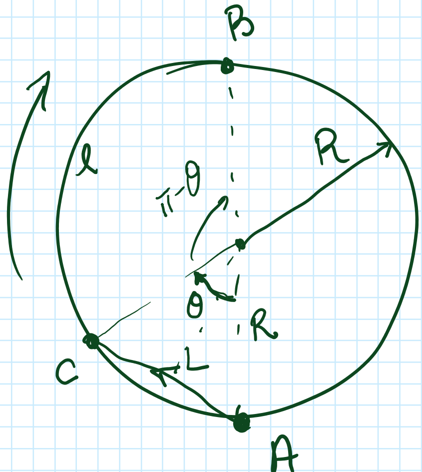


Start from A to arrive at B in least amount of time if swimming speed is known & walking speed is known.

Solution

Stage 1: Notation:

AC Chord length: $L(\theta) = 2R\sin\frac{\theta}{2}$ $\quad \underset{\text{walk}}{0} \le \theta \le \underset{\text{swim}}{\pi}$

CB Arc length: $\ell(\theta) = (\pi-\theta)R$

s : swimming speed $\quad w$: walking speed

Stage 2: Total time $T(\theta) = \frac{L(\theta)}{s} + \frac{\ell(\theta)}{w}$

$$T(\theta) = \frac{2R}{s}\sin\frac{\theta}{2} + \frac{R}{w}(\pi-\theta)$$

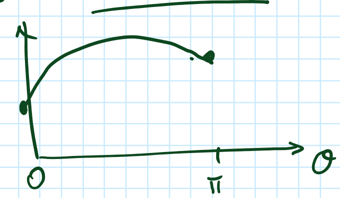
time $\quad \frac{\text{dist.}}{\text{speed}} = \text{time}$ $\quad$ Non-dimensional

Stage 3: Local extrema $T'(\theta) = \frac{R}{s}\cos\frac{\theta}{2} - \frac{R}{w} = 0 \Rightarrow$

$$\cos\frac{\theta}{2} = \frac{s}{w} \Rightarrow \theta_1 = \cos^{-1}\frac{2s}{w}$$

Stage 4: Local min or max

$T''(\theta) = -\frac{R}{2s}\sin\frac{\theta}{2} < 0 \Rightarrow$ local maximum



Stage 5: Check endpoints

$\theta = 0$ (walk only): $T(0) = \frac{\pi R}{w}$

$\theta = \pi$ (swim only): $T(\pi) = \frac{2R}{s}$

Conclusion: combining swimming & walking always leads to longer time

if $T(0) < T(\pi) \Rightarrow \frac{\pi R}{w} < \frac{2R}{s} \Rightarrow w > \frac{\pi}{2}s$

When walking speed is $\left(\frac{\pi}{2}\right) \cong 1.57$ times swimming speed walking minimizes time

if $T(0) = T(\pi)$: $w = \frac{\pi}{2}s$, walking/swimming take the same time

if $T(0) > T(\pi)$: $w < \frac{\pi}{2} s$, swimming takes less time