l'Hôpital

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Antiderivative

$L = \lim_{x\to a} \frac{f(x)}{g(x)}$ $\quad \lim_{x\to a} f(x) = 0 \quad \lim_{x\to a} g(x) = 0$; "$\frac{0}{0}$" indeterminancy

f, g are differentiable $\Rightarrow$

$$L = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

$g(x) \to 0$ as $x \to a$; $\frac{1}{g(x)} \to \pm\infty$ $\qquad G(x) = \frac{1}{g(x)}$

$f(x) \to 0$ — " — $\qquad \frac{1}{f(x)} \to \pm\infty$ $\qquad F(x) = \frac{1}{f(x)}$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{G(x)}{F(x)}$$

"$\frac{0}{0}$" $\qquad$ "$\frac{\infty}{\infty}$" indeterminancy

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l'Hôpital rule is also applicable for $\frac{\infty}{\infty}$ indet.

Ex: $L = \lim_{x\to\infty} \frac{x^4 - 3x^2 + 2x - 1}{2x^4 + 3x^2 - 1}$ $\quad \left(\text{"}\frac{\infty}{\infty}\text{"}\right)$

Method 1 (from Ch. 3) Use algebraic transformation

$$L = \lim_{x\to\infty} \frac{\cancel{x^4}\left(1 - 3\frac{1}{x^2} + 2\frac{1}{x^3} - \frac{1}{x^4}\right)}{\cancel{x^4}\left(2 + 3\frac{1}{x^2} - \frac{1}{x^4}\right)} \Rightarrow$$

$$L = \lim_{x\to\infty} F(x)\cdot G(x) \qquad F(x) = \frac{x^4}{x^4} = 1$$

$$G(x) = \frac{1 - 3\frac{1}{x^2} + 2\frac{1}{x^3} - \frac{1}{x^4}}{2 + 3\frac{1}{x^2} - \frac{1}{x^4}}$$

Recall $\lim_{x\to\infty} F(x)\,G(x) = \left(\lim_{x\to\infty} F(x)\right)\left(\lim_{x\to\infty} G(x)\right) = \frac{1}{2}$

Method 2: l'Hôpital

- $L = \lim_{x\to\infty} \frac{x^4 - 3x^2 + 2x - 1}{2x^4 + 3x^2 - 1} = \lim_{x\to\infty} \frac{4x^3 - 6x + 2}{8x^3 + 6x} =$

- $$L = \lim_{x\to\infty} \frac{\ldots}{2x^4+3x^2-1} = \lim_{x\to\infty} \frac{\ldots}{8x^3+6x} =$$

$$= \lim_{x\to\infty} \frac{12x^2-6}{24x^2+6} = \lim_{x\to\infty} \frac{24\cancel{x}}{48\cancel{x}} = \frac{1}{2}$$

Good idea to maintain factored form of coefficients: $$L = \lim_{x\to\infty} \frac{4\cdot 3x^2-6}{8\cdot 3x^2+6}.$$

Ex: $$L = \lim_{x\to 0} \frac{e^x-x-1}{x^2}$$

$\underbrace{\text{"}\tfrac{0}{0}\text{" indet. ; l'Hôpital}}_{(*)}$ ☺ ☹ $$L = \lim_{x\to 0} \frac{e^x-1}{2x} \overset{☺}{=} \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$

Ex: $$\boxed{L = \lim_{x\to\infty} \frac{e^x}{x^n}} \qquad n\in\mathbb{N}$$

"$\frac{\infty}{\infty}$" indet. $$L = \lim_{x\to\infty} \frac{e^x}{n x^{n-1}} = \ldots = \lim_{x\to\infty} \frac{e^x}{n!} = \infty$$

$\frac{\infty}{2}=\infty \qquad \frac{\infty}{3}=\infty \qquad , \frac{\infty}{\cancel{\text{single}}}=\infty$

e^x increases faster than any positive power of x.

Def: Comparing functions at ∞, $-\infty$

$f(x)$, $g(x)$ $\qquad f, g: \mathbb{R}\to\mathbb{R}$

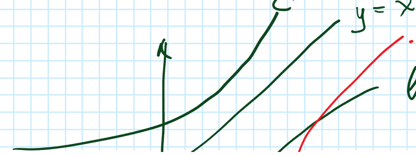
$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = L$ a finite number, non-zero

"f is of the same order as g at ∞"

$f \sim g$ as $x\to\infty$

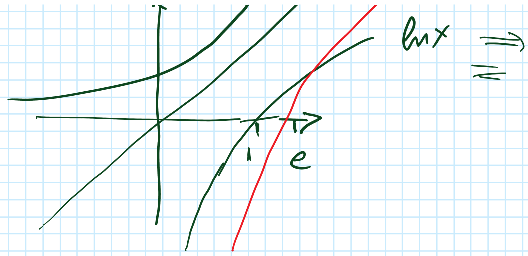
$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \pm\infty$ $\qquad f$ is of higher order than g at $\pm\infty$

$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 0$ $\qquad g$ — " —



Ex: $L = \lim \frac{\ln x}{\ldots}$

Ex: $L = \lim_{x\to\infty} \frac{\ln x}{x} = 0$



$\frac{\infty}{\infty}$ $\quad L = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$

x grows faster than $\ln x$

Ex: $L = \lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2(\ln x)\frac{1}{x}}{1} = 2\lim_{x\to\infty} \frac{\ln x}{x} = 0$

"$\frac{\infty}{\infty}$" indet.

Ex: $L = \lim \frac{(\ln x)^p}{}$

Log. diff.) $\log f = \log\left[(\log x)^x\right] = x \cdot \log(\log x)$

$\frac{d}{dx}$:) $\frac{f'}{f} = 1 \cdot \log(\log x) + x \frac{1}{\log x} \cdot \frac{1}{x}$ (Assume log means natural)

$g(x) = \log(\log x)$; $g'(x) = [u(v(x))]' = u'(v(x))\, v'(x)$

$u(y) = \log y$; $v(x) = \log x$

$u'(y) = \frac{du}{dy} = \frac{1}{y}$; $v'(x) = \frac{1}{x}$

$g'(x) = \frac{1}{\log x} \cdot \frac{1}{x}$

f' $\quad f(x) = (\log x)^x$

$$g'(x) = \frac{\quad}{\log x} \cdot \frac{\quad}{x}$$

$$\frac{f'}{f} = \log(\log x) + \frac{1}{\log x} \; ; \quad f(x) = (\log x)^x$$

$$f' = (\log x)^x \left[\log(\log x) + \frac{1}{\log x} \right]$$

$$f' = \log(\log x)\,(\log x)^x + (\log x)^{x-1}$$

$$L = \lim_{x\to\infty} \frac{(\log x)^x}{x} = \lim_{x\to\infty} \frac{\underline{\log(\log x)\cdot(\log x)^x} + (\log x)^{x-1}}{1} = \infty$$

More indeterminancies for l'Hôpital.

$$\text{"}\frac{0}{0}\text{"}, \quad \text{"}\frac{\infty}{\infty}\text{"}, \quad \text{"}0\cdot\infty\text{"}, \quad \text{"}\infty-\infty\text{"}, \quad 1^\infty, \quad \infty^0$$

$$\text{"}\frac{\infty}{\infty}\text{"} \sim \text{"}\frac{1}{\infty}\cdot\frac{\infty}{1}\text{"} \sim \text{"}\frac{0}{0}\text{"}$$

$$\text{"}0\cdot\infty\text{"} \sim \text{"}0\,\frac{1}{0}\text{"} \; ; \quad \text{"}\infty-\infty\text{"} \sim \text{"}\infty\left(1-\frac{\infty}{\infty}\right)\text{"} \sim$$

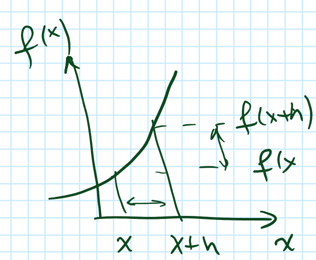
$$\sim \text{"}\frac{1}{0}\;0\text{"}$$

$$\text{"}1^\infty\text{"} \to \text{take logs}$$

$\to$ Memorization cribsheet (backed by Math theorems")

The Antiderivative

$f(x)$ derivative is: $f'(x) = \lim_{h\to} \frac{f(x+h) - f(x)}{h}$



$$\frac{d}{dx} f(x) = f'(x)$$

$f(x)$	$f'(x)$
x^α	$\alpha x^{\alpha-1}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x

$\alpha \in \mathbb{R}$

α could be $\begin{cases} 1, 2, 3 \\ 1.213 \\ \sqrt{\pi + \sqrt{e}} \\ \cancel{\sin x} \end{cases}$

Review

$\cos x$	$\sin x$
e^x	e^x
$\ln x$	$\frac{1}{x}$
a^x	$a^x \ln a$
$\sin^{-1} x = \arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x = \arctan$	$\frac{1}{1+x^2}$

$\Rightarrow$

$[\ \cancel{\sin x}$

Review

Inverse functions

$y = f(x)$

f is one-to-one

$x = f^{-1}(y)$

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{\frac{d}{dx} f(x)}$$

$y = \sin(x)$

$x = \sin^{-1} y$

$$\frac{d}{dy} \sin^{-1} y = \frac{1}{\frac{d}{dx}\sin x} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}$$

Def: (notation) Given some function $f(x)$

The anti-derivative is $F(x)$ such that

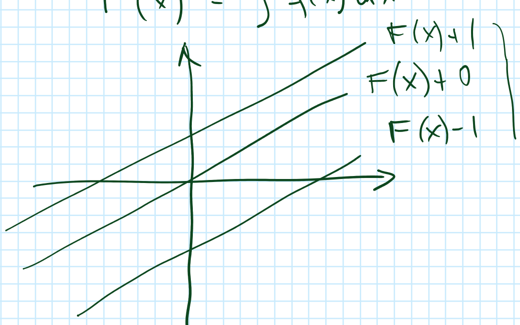
$F'(x) = f(x)$

$$\int f(x)\,dx = F(x) + C \quad \text{with } C \text{ a constant}$$

Recall that $\frac{d}{dx}(c) = 0$

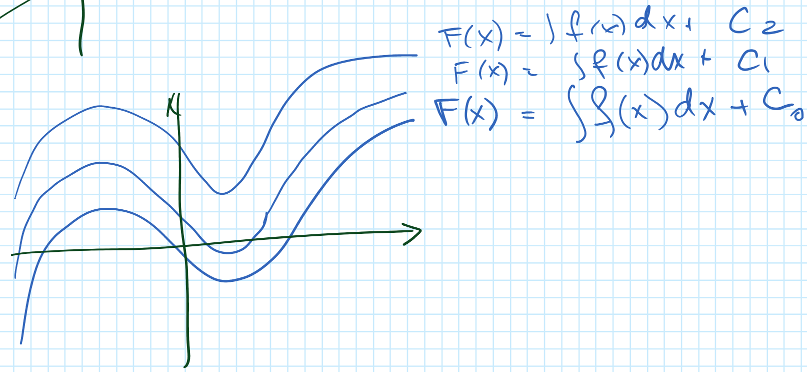
Ex: $f(x) = m$ $\quad$ m a constant.

$$F(x) = \int f(x)\,dx + C = mx + C$$



family of antiderivatives

Ex:



Rules for working with antiderivatives

Sum of antiderivatives $\quad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

Multiplication by a constant $\quad \int c\, f(x)\,dx = c \int f(x)\,dx$

Recall (review) $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

"Derivative of product" $\neq$ "product of derivatives"

Product: $\int f(x)g(x)\,dx \neq \left[\int f(x)\,dx\right]\left[\int g(x)\,dx\right]$

Ex: $\int \sin x\,dx = -\cos x + C$

Ex: $\int (\sin x + x^2)\,dx = \int \sin x\,dx + \int x^2\,dx$
$= -\cos x + \frac{1}{3}x^3 + C$

Ex: $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$; $\frac{d}{dx}\left[\sin^{-1}x + C\right] = \frac{1}{\sqrt{1-x^2}}$ ✓

Ex: $\int \frac{dx}{\sqrt{1-(ax)^2}} = \frac{1}{a}\sin^{-1}ax + C$ $\left(\begin{array}{l} a \in \mathbb{R}, \\ a \text{ constant} \\ a \neq 0 \end{array}\right)$

$\frac{d}{dx}\left[\sin^{-1}ax\right] = \frac{a}{\sqrt{1-(ax)^2}}$

For $a=0$
$\int \frac{dx}{\sqrt{1-0}} = \int dx = x + C$

Def: $\int f(x)\,dx = F(x) + C$

$f(x)$ is the "integrand"

x is the "integration variable"

C —"— "integration constant"

$F(x)$ is the antiderivative

Ex: $I = \int (\sin x)\cdot(\cos x)\,dx = \cancel{(-\cos x)(\sin x) + C}$

Recall that $\sin 2\theta = 2\sin\theta\cos\theta$

$I = \frac{1}{2}\int \sin 2x\,dx = \left(-\frac{1}{4}\right)\cos 2x + C = F(x)$

Check: $F'(x) = \left(-\frac{1}{4}\right)(-2\sin 2x) = \frac{1}{2}\sin 2x = \sin x\cos x$ ✓

Ex: $\int \sin(x+y)\,dx = -\cos(x+y) + C$

Ex: $\int \sin 3x \cos y\,dx = \cos y \int \sin 3x\,dx =$

$= \cos y \left(-\frac{1}{3}\cos 3x\right) + C$

Ex: $\displaystyle\int \sin 3x \cos x \, dx$

Step 1: Recognize product of trig. functs "$(\sin)\cdot(\cos)$"

Step 2: Transform into a sum: of sines

$$\frac{1}{2}(\sin u + \sin v) = \sin\frac{u+v}{2}\cos\frac{u-v}{2}$$

$$\frac{u+v}{2} = 3x \qquad \frac{u-v}{2} = x$$

$$\begin{cases} u+v = 6x \\ u-v = 2x \end{cases}$$

$$2u = 8x \Rightarrow u = 4x$$
$$v = 2x$$

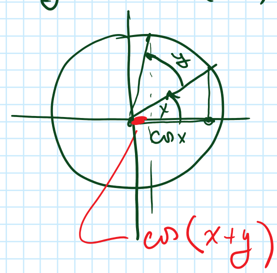
Trig identities

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{y-x}{2}\right)$$



$$\int \sin 3x \cos x \, dx = \frac{1}{2}\int (\sin 4x + \sin 2x)\, dx =$$

$$= \frac{1}{2}\left[\left(-\frac{1}{4}\right)\cos 4x - \frac{1}{2}\cos 2x\right] + C$$

$$= -\frac{1}{4}\left[\frac{1}{2}\cos 4x + \cos 2x\right] + C.$$

Ex: $\displaystyle\int \frac{dx}{x} = \ln x + C$

Ex: $\displaystyle\int e^x \, dx = e^x + C$

Ex: $\displaystyle\int \frac{dx}{x^\alpha} = \int x^{-\alpha}\, dx = \frac{1}{1-\alpha} x^{1-\alpha} + C$ $\quad \alpha \in \mathbb{R}$, α constant

for $\alpha \neq 1$

For $\alpha = 1$ $\quad \displaystyle\int \frac{dx}{x} = \ln x + C$

Further study $\Rightarrow$ approaches to various integrals, e.g.

$\displaystyle\int e^x \sin(mx)\, dx = \ldots$ (MATH 232)

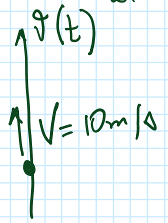
Differential equations: If the slope function is given, what is the function

Initial Value Problem: $\begin{cases} y' = f(x) & \text{finding an antiderivative} \\ & + \\ y(x_0) = y_0 & \text{finding the integration constant} \end{cases}$

Initial Value Problem: $\left\{ \begin{array}{ll} y = f(x) & \text{finding an antiderivative} \\ & + \\ y(x_0) = y_0 & \text{finding the integration constant} \end{array} \right.$

Ex: Motion in a gravitational field

What is the velocity in time of an object thrown upwards at $V = 10\ m/s$?



$$\left.\begin{array}{l} v(t) = \text{velocity} \\ v'(t) = -g \\ v(t) = -gt + C \\ v(0) = 0 + C = V \end{array}\right| \Rightarrow v(t) = V - gt$$