Definite integrals

Thursday, November 10, 2022 9:28 AM

Riemann sum $f: [a,b] \to \mathbb{R}$, partition of $[a,b]$ by
$a = x_0 < x_1 < x_2 < \ldots < x_k < x_{k+1} < \ldots < x_{n-1} < x_n = b$

Left Riemann sum $= \sum_{k=1}^{n} f(x_{k-1}) \Delta_k$ $\quad (\Delta_k = x_k - x_{k-1})$

Right Riemann sum $= \sum_{k=1}^{n} f(x_k) \Delta_k$ $\quad \Delta = \max(\Delta_1, \Delta_2, \ldots, \Delta_k)$

Riemann sum $= \sum_{k=1}^{n} f(x_k^*) \Delta_k$ $\quad x_k^*$ arbitrary $\quad x_{k-1} \le x_k^* \le x_k$

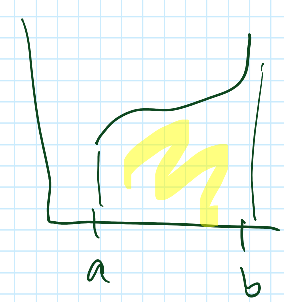
$$\boxed{\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta_k}$$

Δ "Partition size"

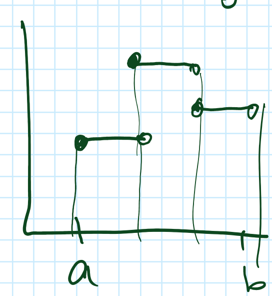
Definite integral of $f(x)$ from a to b

As $\Delta \to 0$, $n \to \infty \Rightarrow$ "$\infty \cdot 0$" indeterminancy

$$\underbrace{\sum f(x_k^*)}_{\substack{\infty \text{ no. of terms} \\ \text{in sum}}} \cdot \underbrace{\Delta_k}_{\downarrow \, 0}$$

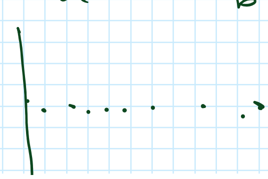


Q1) Does the limit exist?
If f has a finite no. of pts. of discontinuity, and is finite-valued
then $\int_a^b f(x)\,dx$ exists

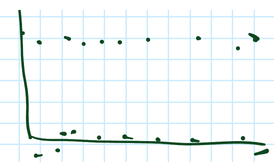


Q2) How do we compute
$\int_a^b f(x)\,dx$?

$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$



$\int_a f(x)\,dx$ $\quad f(x) = \begin{cases} \\ 0 & x \notin \mathbb{Q} \end{cases}$

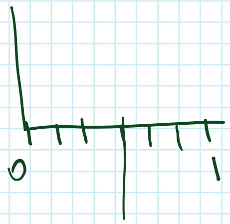


Q2| Can be computed by limits.

Ex: $$\int_0^1 x\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k)\Delta$$

$\Delta = \frac{1}{n}$

$\Delta \to 0$ as $n \to \infty$



$x_k = k\Delta = \frac{k}{n}$

$$= \lim_{\Delta \to 0} \Delta \sum_{k=1}^{n} x_k$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k}{n}$$

$$= \lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n} k$$

$$= \lim_{n \to \infty} \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{1}{2}$$

$P(x) = x^3 + x \qquad T(n) = n^4 - n \qquad S(x) = n^{\sqrt{n}}$

$$\int x\,dx = F(x) = \frac{x^2}{2}$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_0^1 x\,dx = \frac{1}{2} - 0 = \frac{1}{2}$$