Test 1

1.

$$L = \lim_{a\to\infty} \frac{2e^{5x} - 3e^{3a}}{5e^{5x} + 4e^{3a}}$$

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$$L = \lim_{a\to\infty} \frac{\cancel{e^{3a}}}{\cancel{e^{3a}}} \cdot \frac{2e^{5x-3a} - 3}{5e^{5x-3a} + 4} \Rightarrow L = -\frac{3}{4}.$$

e^{5x} constant

$e^{5x-3a} = \frac{e^{5x}}{e^{3a}} \to 0$ as $a \to \infty$

2.

At how many points is the function

$$f(x) = \frac{x^2+1}{x(x^2-3x-4)}$$

discontinuous for $-3 \leqslant x \leqslant 3$?

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$$f(x) = \frac{x^2+1}{x(x-4)(x+1)}$$

Three zeros of denominator $x_1 = 0$, $x_2 = 4$, $x_3 = -1$

but $x_2 \notin [-3,3] \Rightarrow$ 2 pts. of discontinuity

3.

$$L = \lim_{x\to\infty} \left[\sqrt{x^2+ax} - \sqrt{x^2-b}\right].$$

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$A = \sqrt{x^2+ax}$ $\quad B = \sqrt{x^2-b}$ $\quad A^2 - B^2 = x^2 + ax - x^2 + b = ax + b$

$$L = \lim_{x\to\infty} (A-B) = \lim_{x\to\infty} \frac{A^2-B^2}{A+B} = \lim_{x\to\infty} \frac{ax+b}{\sqrt{x^2+ax} + \sqrt{x^2-b}} \Rightarrow$$

$$L = \lim_{x\to\infty} \frac{\cancel{x}\left(a + \frac{b}{x}\right)}{\cancel{x}\left(\sqrt{1+\frac{a}{x}} + \sqrt{1-\frac{b}{x^2}}\right)} = \frac{a}{2}.$$

4.

$$L = \lim_{x\to 0^-} \frac{2}{\tan(x)}.$$

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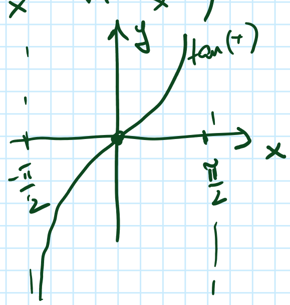

$\tan x = \frac{\sin x}{\cos x}$

as $x \to 0^-$, $\tan x \to 0$

$\tan x < 0$

$\Rightarrow \frac{2}{\tan x} \to -\infty$

$L = -\infty.$

5.

Evaluate and simplify y' for

$$y(t) = 5t^2 + e^{-\ln t}.$$

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$$y(t) = 5t^2 + \frac{1}{e^{\ln t}} = 5t^2 + \frac{1}{t}$$

$$y'(t) = 10t - \frac{1}{t^2}$$

(exp, ln are inverses of one another)

Test 2

1.

$$f(x) = e^{\tan(x)}(\tan(x) - 1)$$

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$f(x) = g(x) \cdot h(x)$; $g(x) = e^{\tan(x)}$; $h(x) = \tan(x) - 1$

$f' = g'h + gh'$; $g'(x) = e^{\tan(x)} \sec^2(x)$

$h'(x) = \sec^2(x)$

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$$h'(x) = \sec^2(x)$$

$$f'(x) = e^{\tan(x)} \sec^2(x)\,(\tan(x)-1) + e^{\tan(x)} \cdot \sec^2(x)$$

$$= e^{\tan(x)} \sec^2(x) \tan(x).$$

2. $f(x) = \ln(\sin(\cos(x)))$;

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$$f(x) = g(h(x)) \qquad g(u) = \ln u \qquad h(x) = \sin(\cos(x))$$

$$g'(u) = \frac{1}{u} \; ; \; h'(x) = \cos(\cos(x))(-\sin(x))$$

$$f'(x) = g'(h(x))\,h'(x) = \frac{1}{\sin(\cos(x))} \cos(\cos(x))\,(-\sin(x)) \Rightarrow$$

$$f'(x) = -\cot(\cos(x))\sin(x).$$

3. $y(x)$

$xy^4 + x^4 y = 1.$

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$$\frac{d}{dx}\left(xy^4 + x^4 y\right) = \frac{d}{dx}(1) \Rightarrow$$

$$y^4 + 4xy^3y' + 4x^3y + x^4y' = 0 \Rightarrow$$

$$x(4y^3 + x^3)\,y' = -y(y^3 + 4x^3) \Rightarrow y' = -\frac{y}{x} \cdot \frac{y^3 + 4x^3}{4y^3 + x^3}$$

4. $y(x) = \dfrac{x^8 \cos^3(x)}{\sqrt{x-1}}.$

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Logarithmic differentiation:

$$\ln y = 8 \ln x + 3 \ln \cos(x) - \frac{1}{2} \ln(x-1)$$

Take $\frac{d}{dx}$ $\Rightarrow$

$$\frac{y'}{y} = \frac{8}{x} - 3\tan(x) - \frac{1}{2}\,\frac{1}{x-1} \Rightarrow$$

$$y' = \left(\frac{8}{x} - 3\tan(x) - \frac{1}{2}\,\frac{1}{x-1}\right)\frac{x^8\cos^3(x)}{\sqrt{x-1}}.$$

5. $f(x) = 6x^4 \ln(x^2) - 7x^4.$

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Local min/max are solutions of $f'(x)=0$ with $f''(x) \neq 0$. Note that $f: (0,\infty) \to \mathbb{R}$

$$f(x) = 12x^4 \ln x - 7x^4 = x^4(12\ln x - 7)$$

$$f'(x) = 4x^3(12\ln x - 7) + x^4\left(\frac{12}{x}\right) = x^3\left[4(12\ln x - 7) + 12\right]$$

$f'(x) = 0 \Rightarrow x = 0$ (inadmissible since f not defined at 0)

or

$$4(12\ln x - 7) + 12 = 0 \Rightarrow 12\ln x - 7 = -3$$

$f'(x) = 0 \Rightarrow x = 0$ (inadmissible since ...
or

$4(12\ln x - 7) + 12 = 0 \Rightarrow 12\ln x - 7 = -3$

$\Rightarrow 12\ln x = 4 \Rightarrow \ln x = \frac{1}{3} \Rightarrow x = e^{\frac{1}{3}}$

$f''(x) = 3x^2\left[4(12\ln x - 7) + 12\right] + x^3 \frac{48}{x}$

$f''(e^{\frac{1}{3}}) = 3e^{\frac{2}{3}}\left[4\left(12\cdot\frac{1}{3} - 7\right) + 12\right] + 48e^{\frac{2}{3}}$

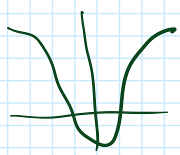
$= 3e^{\frac{2}{3}}\left[-12 + 12\right] + 48e^{2/3} = 48e^{2/3} > 0 \Rightarrow x = e^{\frac{1}{3}}$ is a min.

Test 3

1. $y(x) = \tan^{-1}(x^2)$ $\quad y'(x) = \frac{2x}{1+(x^2)^2} = \frac{2x}{1+x^4}$

$y'(0) = 0$; $\quad y'(x) < 0$ for $x < 0 \Rightarrow y$ decreases
$\quad y'(x) > 0$ for $x > 0 \Rightarrow y$ increases $\Rightarrow$

(a)



2. $y(0.25)$ for $y(x) = x + \ln(1+x)$; $y'(x) = 1 + \frac{1}{1+x}$

Linear approximant $\quad L(x) = y'(a)(x-a) + y(a)$ $\Rightarrow$

Choose $a = 0 \Rightarrow L(x) = 2x$

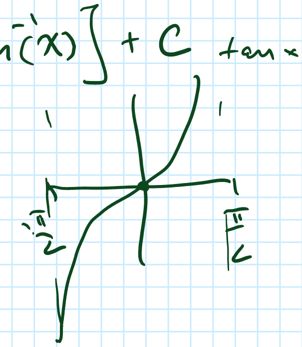
$y(0.25) \cong L(0.25) = 0.5 \Rightarrow$ (b) $y = 0.25$ is closest.

3. $f(x) = \frac{5x^2}{4+4x^2}$; $\quad F(x) = \int f(x)\,dx$

$F(x) = \frac{5}{4}\int \frac{x^2\,dx}{1+x^2}$ $\quad$ Recall $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$\Rightarrow F(x) = \frac{5}{4}\int\left(\frac{x^2+1-1}{1+x^2}\right)dx = \frac{5}{4}\int\left(1 - \frac{1}{1+x^2}\right)dx \Rightarrow$

$F(x) = \frac{5}{4}\left[\int dx - \int\frac{dx}{1+x^2}\right] = \frac{5}{4}\left[x - \tan^{-1}(x)\right] + C$



4. $L = \lim\limits_{x\to\infty} \frac{\tan^{-1}(x) - \frac{\pi}{2}}{1/x}$ $\quad \lim\limits_{x\to\infty}\tan^{-1}(x) = \frac{\pi}{2}$

$\Rightarrow$ "$\frac{0}{0}$" indeterminancy $\Rightarrow$ l'Hôpital $\Rightarrow$

$L = \lim \quad \frac{\frac{1}{1+x^2}}{\ldots} = -\lim \frac{x^2}{\ldots} = -1$

$$L = \lim_{x \to \infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}} = -\lim_{x \to \infty} \frac{x^2}{x^2+1} = -1$$

5. $f(x) = \dfrac{3x^3+5}{x}$ (that passes through $(1,3)$)

$$F(x) = \int f(x)\,dx = \int \left(3x^2 + \frac{5}{x}\right) dx = \int 3x^2\,dx + 5\int \frac{dx}{x} \Rightarrow$$

$$F(x) = x^3 + 5\ln x + C$$

Pass through $(1,3) \Rightarrow \quad F(1) = 3 \Rightarrow$

$$1 + C = 3 \Rightarrow C = 2 \qquad \Bigg| \Leftrightarrow$$

$$F(x) = x^3 + 5\ln x + 2.$$