Solutions

Tuesday, November 22, 2022 9:30 AM

These are the correct solutions for "egg-hunting" E.C.

Test 1

1. The function

$$f(x) = \sqrt{\frac{x-1}{x-3}}$$

a) has a defined limit as $x \to 1$.

b) has a defined limit as $x \to 1^+$.

c) has a defined limit as $x \to 1^-$. $f: (-\infty, 1] \to \mathbb{R}$, $\lim_{x \to 1^-} f(x) = 0$

d) does not have a limit.

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$\frac{x-1}{x-3} \geq 0$ to take square root

x	$-\infty$			1			3			∞
$x-1$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$		
$x-3$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$
$\frac{x-1}{x-3}$	$+$	$+$	$+$	0	$-$	$-$	$\pm\infty$	$+$	$+$	$+$

$\Rightarrow f: (-\infty, 1] \cup (3, \infty)$

2. How many asymptotes (horizontal, vertical and slant) does the function

$$f(x) = \frac{2x^2+6}{2x^2+3x-2}$$

have?

a) None

b) 4 $x \to -\infty, y = 1$, $x \to \infty, y = 1$, $x \to \frac{1}{2}$ $y \to \pm\infty$, $x \to -2$ $y \to \pm\infty$

c) 2

d) 1

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$\lim_{x \to \pm\infty} f(x) = 1 \Rightarrow y = 1$ horizontal asymptotes (at $+\infty$, $-\infty$)

$2x^2 + 3x - 2 = 0 \Rightarrow$

$(2x-1)(x+2) = 0 \Rightarrow x = -2$, $x = \frac{1}{2}$ } Vertical asymptotes

3. Determine the limit

$$L = \lim_{x \to 3} \frac{\sqrt{3x+16} - 5}{x-3}$$

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1) Direct substitution gives $\frac{\sqrt{9+16} - 5}{3-3} = \frac{0}{0}$ indeterminancy

2) Rewrite $\frac{\sqrt{3x+16} - 5}{x-3} = \frac{(\sqrt{3x+16} - 5)(\sqrt{3x+16} + 5)}{(x-3)(\sqrt{3x+16} + 5)} = \sim$

$\sim = \frac{3x+16-25}{(x-3)(\sqrt{3x+16}+5)} = \frac{3\cancel{(x-3)}}{\cancel{(x-3)}(\sqrt{3x+16}+5)} = \sim$

$\sim = \frac{3}{\sqrt{3x+16}+5} \Rightarrow L = \frac{3}{10}$

4.

4. Determine the limits at $\pm\infty$ of the function

$$f(x) = \frac{e^{-x} + 7x^e}{2e^{-x} + 4x^{2e}}.$$

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$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^e (e^{-x} x^{-e} + 7)}{2x^{2e} (e^{-x} x^{-2e} + 2)}$$

As $x \to \infty$, $e^{-x} \to 0$, $x^{-e} = \frac{1}{x^e} \to 0$, $x^{-2e} = \frac{1}{x^{2e}} \to 0$

Hence $\lim_{x \to \infty} f(x) = 0$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{\cancel{e^{-x}} (1 + 7x^e e^x)}{\cancel{e^{-x}} (2 + 4x^{2e} e^x)} = \frac{1}{2}$$

since as $x \to -\infty$

$x^e e^x \to 0$

$x^{2e} e^x \to 0$

5.

$u = ...$ Product rule $(fg)' = f'g + fg'$

5.

5. Evaluate and simplify y' for

$$y(x) = 5t^2 e^{-t} \sin t.$$

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$y = fg$. Product rule $(fg)' = f'g + fg'$

$f(t) = 5t^2$, $f'(t) = 10t$

$g(t) = e^{-t} \sin t = u(t) v(t)$ $\Rightarrow$ $\Rightarrow$

$g' = u'v + uv'$

$y'(x) = \frac{dy}{dx} = \frac{d}{dx}(5t^2 e^{-t} \sin t) = 0$

$u(t) = e^{-t}$, $u'(t) = -e^{-t}$, $v(t) = \sin t$, $v'(t) = \cos t$

$g'(t) = -e^{-t} \sin t + e^{-t} \cos t = e^{-t}(\cos t - \sin t)$

$y'(t) = 10t e^{-t} \sin t + 5t^2 e^{-t}(\cos t - \sin t).$

Test 2

1.

1. What is the derivative of the function

$$f(x) = \frac{\sin(x) + \cos(y)}{\cos(x) + \sin(y)}?$$

a) $f'(x) = \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{(\cos(x) + \sin(y))^2}$.

b) $f'(x) = \frac{\sin(x)\cos(x) + \cos(y)\sin(y) - 1}{(\cos(x) + \sin(y))^2}$.

c) $f'(x) = \frac{\sin(x)\cos(y) + \cos(x)\sin(y) + 1}{(\cos(x) + \sin(y))^2}$.

d) $f'(x) = \frac{\sin(x)\cos(y) + \cos(x)\sin(y) + 1}{(\cos(x))^2 + (\sin(y))^2}$.

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$f = \frac{u}{v}$ $\quad f' = \frac{u'v - uv'}{v^2}$

$u(x) = \sin x + \cos y$

$u'(x) = \cos x$

$v(x) = \cos x + \sin y$

$v'(x) = -\sin x$

$u'(x)v(x) - u(x)v'(x) = \cos x(\cos x + \sin y)$

$- (\sin x + \cos y)(-\sin x) =$

$= \cos^2 x + \sin^2 x + \cos x \sin y + \sin x \cos y$

$= 1 + \cos x \sin y + \sin x \cos y$

2.

2. Identify the three functions f, g, h in the composite function

$$u(t) = (f \circ g \circ h)(t) = f(g(h(t))) = \cos^4(t^2 + 1).$$

a) $f(x) = t^4$, $g(x) = \cos(t)$, $h(t) = t^2 + 1$.

b) $f(x) = x^4$, $g(x) = \cos(x)$, $h(x) = x^2 + 1$.

c) $f(t) = \cos(t)$, $g(t) = t^4$, $h(t) = t^2 + 1$.

d) $f(u) = u^4$, $g(v) = \cos(v)$, $h(x) = t^2 + 1$.

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a) In $f(x) = t^4$ t is a parameter, not the independent variable.

Introduce z, $u(z) = f(g(h(z))) = \cos^4(z^2+1)$

$h(z) = z^2 + 1$; $g(h(z)) = g(z^2+1) = \cos(t)$

$f(g(h(z))) = t^4 \Rightarrow (f \circ g \circ h)(z) = t^4$ ✗

b) $h(x) = x^2 + 1 \rightarrow h(t) = t^2 + 1$

$g(x) = \cos(x) \Rightarrow g(h(t)) = \cos(t^2+1)$

$f(x) = x^4 \Rightarrow f(\cos(t^2+1)) = \cos^4(t^2+1)$ ✓

c) $(f \circ g \circ h)(t) = \cos((t^2+1)^4)$ ✗

d) $(f \circ g \circ h)(z) = f(g(h(z))) = f(g(t^2+1)) = f(\cos(t^2+1)) = \cos^4(t^2+1)$

$\Rightarrow (f \circ g \circ h)(z) = \cos^4(t^2+1)$ ✗

t not the independent variable

$\Rightarrow \quad (f \circ g \circ h)(z) = \cos(t+1)$ ✗

t not the independent variable

3.

3. Find the derivative $y'(x)$ of the function $y(x)$ defined implicitly by

$$\cos(y^2) + x = a^y, a > 0.$$

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Take derivative $\frac{d}{dx}$

$$\frac{d}{dx}\left[\cos(y^2) + x\right] = \frac{d}{dx}\left[a^y\right] \Rightarrow$$

$$-\sin(y^2)(2y)y' + 1 = a^y \cdot y' \ln a \Rightarrow$$

$$y'\left(a^y \ln a + 2y \sin y^2\right) = 1 \Rightarrow \quad y' = \frac{1}{a^y \ln a + 2y \sin y^2}$$

4.

4. Determine the critical points of

$$f(x) = 10^x\left(\ln(10^x) - x\right).$$

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Compute $f'(x)$

$f = uv$; $f' = u'v + uv'$ $\Rightarrow$

$u = 10^x$, $u' = 10^x \ln 10$, $v = \ln(10^x) - x$

$v = x \ln 10 - x$, $v' = \ln 10 - 1$

$$f'(x) = x 10^x \cdot \ln 10 \cdot (\ln 10 - 1) + 10^x(\ln 10 - 1) \Rightarrow$$

$$f'(x) = 10^x \cdot (\ln 10 - 1) \cdot (x \ln 10 + \cancel{10^x}).$$

$$10^x \cdot (\ln 10 - 1) \cdot (x \ln 10 + 1)$$

5. :

5. An object is moving vertically according to the distance function $s(t) = t^3 - 8t^2 - 12t - 3$. Determine the intervals over which the object is moving down.

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Object moving down $\Rightarrow$ velocity is negative.

Velocity $v(t) = s'(t) = 3t^2 - 16t - 12 = (3t+2)(t-6)$

t		$-\frac{2}{3}$		6	
$3t+2$	$-$ $-$ $-$	0	$+$ $+$	$+$	$+$
$t-6$	$-$	$-$	$-$	0	$+$ $+$ $+$
$(t-6)(3t+2)$	$+$ $+$	0	$-$ $-$	0	$+$ $+$

Object moves down for $-\frac{2}{3} < t < 6$.

Test 3

1.

1. What is the linear approximation $L(x)$ of $f(x) = x\ln(x) + 1$ near $x = 1$?

a) $L(x) = \ln(x) + 1$. Not linear

b) $L(x) = x\ln(x)$. Not linear

c) $L(x) = 1 - x$. Wrong sign

d) $L(x) = x$.

$L(x) = f'(1)(x-1) + f(1)$

$f(1) = 1 \cdot \ln 1 + 1 = 1$

$f'(x) = \ln x + 1 \Rightarrow f'(1) = 1$

$\Rightarrow L(x) = x - 1 + 1 = x$

c) $L(x) = 1 - x$. wrong sign

d) $L(x) = x$.

$\Rightarrow L(x) = x - 1 + 1 = x$

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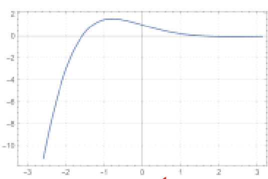
2.

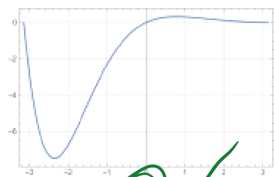
2. What is the plot of the function $f(x) = e^{-x}\sin(x)$? Provide a brief motivation.



~~(a)~~

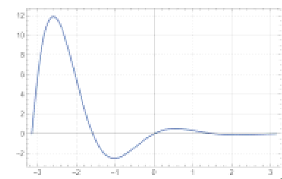
Why, or why not?

At $x=0$, f must be zero



$-\pi$ (b) ✓ π

Why, or why not?



$-\pi$ ~~(c)~~ π

Why, or why not?

At $x = -\pi$, $f' < 0$

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$f(0) = 0$

$f'(x) = -e^{-x}\sin x + e^{-x}\cos x$

$f'(\pi) = -e^{-\pi}\cdot 0 + e^{-\pi}\cdot(-1) < 0$

3.

3. Find the anti-derivative $F(x)$ of

$$f(x) = \frac{6}{\sqrt{4-4x^2}}.$$

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$g(x) = \sin^{-1} x \Rightarrow x(y) = \sin y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$F(x) = \int f(x)\,dx = 3\int \frac{dx}{\sqrt{1-x^2}} = 3\sin^{-1} x + C$$

4.

4. Find the anti-derivative $F(x)$ of

$$f(x) = \frac{4}{x\sqrt{x^2-1}}.$$

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Problem suggests substitution

$u(x) = \sqrt{x^2-1} \Rightarrow u^2 = x^2 - 1 \Rightarrow x^2 = 1 + u^2$

$du = \frac{x}{\sqrt{x^2-1}}\,dx \Rightarrow dx = \frac{\sqrt{x^2-1}}{x}\,du$

$$F(x) = \int f(x)\,dx = \int \frac{4}{x\sqrt{x^2-1}}\frac{\sqrt{x^2-1}}{x}\,du = 4\int \frac{du}{x^2} = 4\int \frac{du}{1+u^2} \Rightarrow$$

$$F(x) = 4\tan^{-1} u + C = 4\tan^{-1}\left(\sqrt{x^2-1}\right) + C.$$

5.

5. Compute the limit

$$L = \lim_{x\to 2\pi} \frac{x\sin(x) + x^2 - 4\pi^2}{x - 2\pi}.$$

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As $x \to 2\pi$ obtain $\frac{2\pi\cdot 0 + 4\pi^2 - 4\pi^2}{2\pi - 2\pi} = \frac{0}{0}$ indet.

Apply l'Hôpital: $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$

$$L = \lim_{x\to 2\pi} \frac{\sin x + x\cos x + 2x}{1} = 2\pi + 4\pi = 6\pi.$$

$$L = \lim_{x \to 2\pi} \frac{\sin x + x\cos x + 2x}{1} = 2\pi + 4\pi = 6\pi .$$