Practice Test 2 - Solution

1. Compute the derivative $s'(t)$ of

$$s(t) = \frac{t^{4/3}}{e^t}$$

exists, and, if so, compute the limit.

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Write $s(t) = \frac{f(t)}{g(t)}$ with $f(t) = t^{4/3}$, $g(t) = e^t$

Quotient rule: $s' = \frac{f'g - fg'}{g^2}$

$f'(t) = \frac{4}{3}t^{1/3}$ $g'(t) = e^t$ $\Rightarrow$

$$s'(t) = \frac{\frac{4}{3}t^{1/3}e^t - t^{4/3}e^t}{e^{2t}} = \frac{t^{1/3}(4-3t)}{3e^t}$$

Since $g(t) \neq 0$, $g'(t) = 0$: derivative exists for all $t \in (-\infty, \infty)$

2. Write $y(x) = \frac{u(x)}{v(x)}$ with $u(x) = a\sin x + b\cos x$ $\quad v(x) = a\sin x - b\cos x$

$u'(x) = a\cos x - b\sin x$ $\quad v'(x) = a\cos x + b\sin x$

2. Compute the derivative $y'(x)$ of

$$y(x) = \frac{a\sin x + b\cos x}{a\sin x - b\cos x}$$

where a, b are nonzero constants..

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Quotient rule: $y' = \frac{u'v - uv'}{v^2} \Rightarrow$

Compute $u'v - uv'$ with notation $s = \sin x$; $c = \cos x$

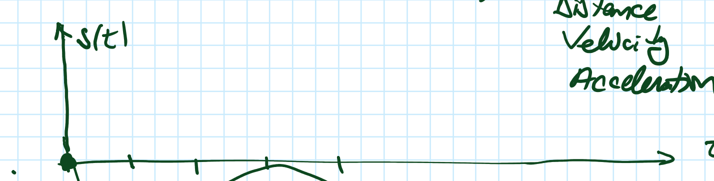
$$u'v - uv' = (ac - bs)(as - bc) - (as + bc)(ac + bs) =$$

$$= \cancel{a^2cs} - abs^2 - abc^2 + \cancel{b^2sc} - \cancel{a^2sc} - abc^2 - abs^2 - \cancel{b^2cs} =$$

$$= 2ab(s^2 + c^2) = 2ab(\cos^2 x + \sin^2 x) = 2ab$$

$$\Rightarrow \quad y' = \frac{2ab}{(a\cos x + b\sin x)^2}$$

3.



$s(t) = -6t(t^2 - 6t + 9)$

Distance $s(t) = -6t^3 + 36t^2 - 54t$ $\quad s(0) = 0$

Velocity $s'(t) = -18t^2 + 72t - 54$ $\quad s(0) = -54$

Acceleration $s''(t) = -36t + 72$ $\quad s(0) = 72$

t	0		1		2		3		4
s	0	↘	−24	↗		↗	0	↘	
s'	−54	−	0	+	+	+	0	−	−
s''	72	+	+	36 +	0	−	−36	−	

3. The distance of an object from its starting point is $s(t) = -6t^3 + 36t^2 - 54t$. Determine the acceleration of the object whenever its velocity is zero for $0 \leqslant t \leqslant 4$.

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zero velocity $\Rightarrow$ $s'(t) = 0$ $\Rightarrow$ $18t^2 - 72t + 54 = 0 \Leftrightarrow 9t^2 - 36t + 27 = 0$

Observe $s'(1) = 0$, hence $9t^2 - 36t + 27 = (t-1)(9t - 27)$ with roots

$t_1 = 1$, $t_2 = 3$ $\quad 0 \leq t_1, t_2 \leq 4$

Acceleration at these times: $s''(t_1) = -36 + 72 = 36$

$s''(t_2) = -108 + 72 = -36$

4.

4. Consider $y(x)$ defined implicitly by

$$(x+y)^{2/3} = y.$$

$$\frac{d}{dx}y = \frac{d}{dx}(x+y)^{2/3}$$

(diff. of powers $\Rightarrow$) & chain rule

$$y' = \frac{2}{3}(x+y)^{-1/3}(1 + y') \Rightarrow$$

4. Consider $y(x)$ defined implicitly by

$$(x+y)^{2/3} = y.$$

Compute $y'(x)$ and the slope of $y(x)$ at $(x,y) = (4,4)$.

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& chain rule

$$y' = \frac{2}{3}(x+y)^{-1/3}(1+y') \Rightarrow$$

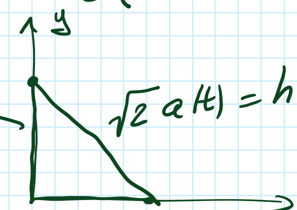
$$\left[1 - \frac{2}{3(x+y)^{1/3}}\right] y' = \frac{2}{3(x+y)^{1/3}} \Rightarrow$$

$$y' = \frac{2}{3(x+y)^{1/3} - 2};$$

At (4,4) $y'(4) = \dfrac{2}{3(4+4)^{1/3} - 2} = \dfrac{1}{2}$.

5.

length of other leg $= a(t)$



$\sqrt{2}\, a(t) = h(t)$ hypotenuse

$a(t)$ - length of one leg

5. The hypotenuse of an isosceles right triangle decreases in length at rate $4m/s$.

a) At what rate is the area of the triangle increasing when the legs are $5m$ long?

b) At what rate are the lengths of the legs of the triangle changing?

c) At what rate is the area of the triangle changing when the area is $4m^2$?

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$h'(t) = 4\ \frac{m}{s} \Rightarrow a'(t) = \frac{4}{\sqrt{2}}$

Area: $A(t) = \frac{1}{2} a^2(t)$

$A'(t) = a'(t)\, a(t)$

a) when $a(t) = 5$ $\quad A'(t) = \frac{4}{\sqrt{2}} 5 = \frac{20}{\sqrt{2}} = 10\sqrt{2}\ \frac{m^2}{s}$

b) $h(t) = \sqrt{2}\, a(t) \Rightarrow a'(t) = 4/\sqrt{2}\ \frac{m}{s}$

c) A' when $A(t) = 4 = \frac{1}{2} a^2 \Rightarrow a = \sqrt{8} = 2\sqrt{2}$

$A'(t) = \frac{4}{\sqrt{2}} 2\sqrt{2} = 8\ \frac{m^2}{s}$