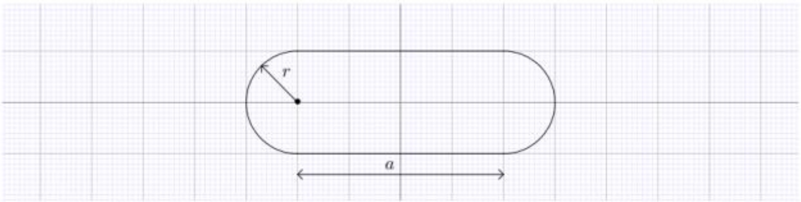
Practice Test 3 Solution

Sunday, November 13, 2022 6:06 PM

Solve the following exercises explaining all solution steps. (45 minutes)

1. Determine the maximal area of two-semicircles and a rectangle that can be enclosed within a known perimeter P.



2. Construct the linear approximant of $f(x) = \ln(1+x)$ at $a = 0$.

3. Determine the limit
$$L = \lim_{c\to 3} \frac{c-1-\sqrt{c^2-5}}{c-3}.$$

4. Determine the limit
$$L = \lim_{x\to\infty} \frac{\ln(3x+5e^x)}{\ln(7x+3e^{2x})}.$$

5. Evaluate the integral
$$I = \int \frac{e^{2x}-5e^x+4}{e^x-1}\,dx.$$

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1. Total area $= A = \pi r^2 + 2ar$

Perimeter $= P = 2\pi r + 2a$

r: ind. var; $A(r)$

$A(r) = \pi r^2 + (P - 2\pi r)r$

Crit. pts. $A'(r) = 0 \Rightarrow$

$2\pi r + P - 4\pi r = 0 \Rightarrow$ with solution

$r_1 = \dfrac{P}{2\pi}$

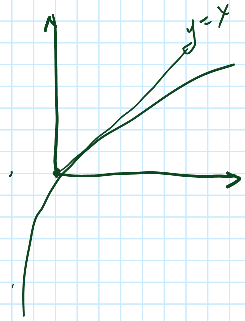
$A''(r) = -2\pi < 0 \Rightarrow r_1$ is a maximum.

$a_1 = \left(P - 2\pi \dfrac{P}{2\pi}\right)\dfrac{1}{2} = 0$

$\Rightarrow$ Entire area enclosed in a circle

$A(r_1) = \pi \left(\dfrac{P}{2\pi}\right)^2 = \dfrac{P^2}{4\pi}$.

2.



$f(x) = \ln(1+x)$; $f'(x) = \dfrac{1}{1+x}$; $f'(a) = \dfrac{1}{1+0} = 1$

$f(a) = \ln 1 = 0 \Rightarrow$

Linear approx. $y = f'(a)(x-a) + f(a) \Rightarrow$

$\boxed{y = x}$

3. $L = \lim\limits_{c\to 3} \dfrac{f(c)}{g(c)}$; As $c \to 3$ $f(c) = c - 1 - \sqrt{c^2-5} \to 3 - 1 - \sqrt{9-5} \to 0$, $g(c) = c - 3 \to 0$ $\Rightarrow$ "$\dfrac{0}{0}$" indeterm.

Apply l'Hôpital $L = \lim\limits_{c\to 3} \dfrac{f'(c)}{g'(c)} = \lim\limits_{c\to 3} \dfrac{1 - \frac{c}{\sqrt{c^2-5}}}{1} = 1 - \lim\limits_{c\to 3} \dfrac{c}{\sqrt{c^2-5}} = -\dfrac{1}{2}$.

4. $L = \lim\limits_{x\to\infty} \dfrac{\ln(3x+5e^x)}{\ln(7x+3e^{2x})} = \lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)}$, "$\dfrac{\infty}{\infty}$" indeterminancy, apply l'Hôpital $\Rightarrow$

$L = \lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ $\quad f(x) = \ln(3x+5e^x)$; $f'(x) = \dfrac{3+5e^x}{3x+5e^x}$ (chain rule)

$g(x) = \ln(7x+3e^{2x})$; $g'(x) = \dfrac{7+6e^{2x}}{7x+3e^{2x}}$ (— " —)

$L = \lim\limits_{x\to\infty} \dfrac{(3+5e^x)(7x+3e^{2x})}{(3x+5e^x)(7+6e^{2x})} = \lim\limits_{x\to\infty} \dfrac{21x + 9e^{2x} + 35xe^x + 15e^{3x}}{21x + 35e^x + 18xe^{2x} + 30e^{3x}} \Rightarrow$

$$L = \lim_{x\to\infty} \frac{\cdots}{(3x+5e^x)(7+6e^{2x})} = \lim_{x\to\infty} \frac{\cdots}{21x + 35e^x + 18xe^{2x} + 30e^{3x}}$$

$$L = \lim_{x\to\infty} \frac{\cancel{e^{3x}}}{\cancel{e^{3x}}} \frac{(21xe^{-3x} + 9e^{-x} + 35xe^{-2x} + 15)}{(21xe^{-3x} + 35e^{-2x} + 18xe^{-x} + 30)} = \frac{15}{30} = \frac{1}{2}.$$

since $e^{-3x}, e^{-x}, e^{-2x} \to 0$ as $x \to \infty$

5. $$I = \int \frac{e^{2x} - 5e^x + 4}{e^x - 1}\,dx$$

Let $e^x = t \Rightarrow \dfrac{e^{2x} - 5e^x + 4}{e^x - 1} = \dfrac{t^2 - 5t + 4}{t-1} = \dfrac{(t-1)(t-4)}{t-1} = t - 4 = e^x - 4$

$$\Rightarrow I = \int (e^x - 4)\,dx = e^x - 4x + C.$$