R04 example solutions

Wednesday, September 14, 2022 2:45 PM

3.5.47 Calculate derivative of $f(x) = \dfrac{\cot x}{1+\csc x}$

Function has form $f(x) = \dfrac{g(x)}{h(x)}$ $\quad g(x) = \cot x$, $h(x) = 1+\csc x$

Introduce notation to carry out differentiation of quotient

Apply differentiation of quotient rule

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h^2(x)}$$

Compute derivatives $g'(x) = \dfrac{d}{dx}\cot x = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right)$

Apply differentiation of quotient again

by quotient rule

$$g'(x) = \frac{(\cos x)'\sin x - \cos x\,(\sin x)'}{\sin^2 x} = \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$

$$g'(x) = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

Compute derivatives of the functions

Apply trigonometric identity $\sin^2 x + \cos^2 x = 1$

$$g'(x) = -\frac{1}{\sin^2 x}; \quad h'(x) = \frac{d}{dx}(1+\csc x)$$

Apply differentiation of sum rule

$$h'(x) = \left(\frac{d}{dx}1\right) + \frac{d}{dx}(\csc x) = 0 + \frac{d}{dx}(\csc x)$$

$$h'(x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

Apply derivative of quotient rule

$$h'(x) = \frac{(1)'\cdot\sin x - 1\cdot(\sin x)'}{\sin^2 x} = \frac{0 - \cos x}{\sin^2 x} = -\frac{\cos x}{\sin^2 x}$$

$$h'(x) = -\frac{\cos x}{\sin x}\,\frac{1}{\sin x} = -\cot x \cdot \csc x$$

Replace in

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h^2(x)} = -\frac{\frac{1+\csc x}{\sin^2 x} + \cot^2 x \csc x}{(1+\csc x)^2}$$

$$f'(x) = \frac{-\frac{1+\csc x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\frac{1}{\sin x}}{} = \frac{-(1+\csc x)\sin x + \cos^2 x}{} =$$

$$f'(x) = \frac{-\frac{1+\csc x}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\frac{1}{\sin x}}{(1+\csc x)^2} = \frac{-(1+\csc x)\sin x + \cos^2 x}{\sin x \sin^2 x (1+\csc x)^2} =$$

Carry out this identity simplifications

$$= \frac{-(\sin x + 1) + \cos^2 x}{\sin x \cdot (\sin x + 1)^2} = \frac{-\sin x + \cos^2 x - 1}{\sin x (\sin x + 1)^2} =$$

$$= \frac{-\sin x - \sin^2 x}{\sin x (\sin x + 1)^2} = -\frac{\sin x + 1}{(\sin^2 x + 1)^2} = -\frac{1}{1+\sin x}$$

Verification in Wolfram Alpha.

(Differentiation is well-suited to computer evaluation

D[Cot[x]/(1+Csc[x]),x]

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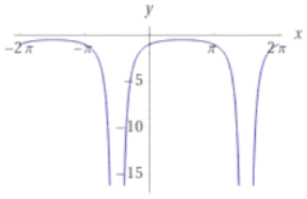
Derivative

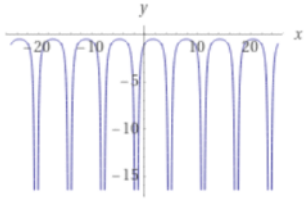
$$\frac{d}{dx}\left(\frac{\cot(x)}{1+\csc(x)}\right) = -\frac{\csc(x)\left(-\cot^2(x) + \csc^2(x) + \csc(x)\right)}{(1+\csc(x))^2}$$

← Form before applying this identities

Plots



(x from −6.283 to 6.283)



(x from −25.13 to 25.13)

Alternate forms

$$-\frac{1}{\sin(x)+1}$$

← Form after applying this identities.

Matches previous calculation!

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