R10 – Model Solution (4.5 Optimization)

32. Circle and square A piece of wire of length 60 is cut, and the resulting two pieces are formed to make a circle and a square. Where should the wire be cut to (a) maximize and (b) minimize the combined area of the circle and the square?

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4.5.32.

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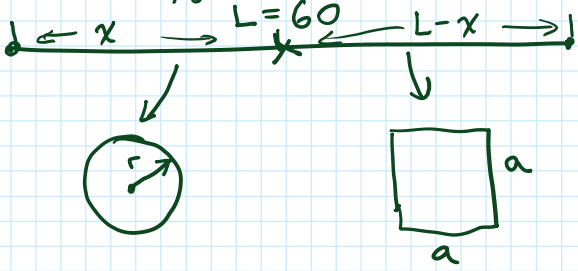
Stage 1: Sketch figure, introduce notation



Stage 2: State equations.

Area of circle: $A(r) = \pi r^2$ Circumference of circle: $C = 2\pi r$

—11— square: $S(a) = a^2$ Perimeter of square: $P = 4a$

Length x of wire forms circle: $2\pi r = x \Rightarrow r = \frac{x}{2\pi}$

Length $L-x$ of wire forms square: $4a = L-x \Rightarrow a = \frac{L-x}{4}$

Total area: $T(x) = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{L-x}{4}\right)^2 = \frac{x^2}{4\pi} + \frac{1}{16}(L-x)^2$

Stage 3: Find local extrema

$$T'(x) = \frac{x}{2\pi} - \frac{L-x}{8} = 0 \Rightarrow 4x = \pi(L-x) \Rightarrow$$

$$x_1 = \frac{\pi L}{\pi + 4}$$

Stage 4: Check if x_1 extremum is a local maximum or minimum

$$T''(x) = \frac{1}{2\pi} + \frac{1}{8} > 0 \Rightarrow x_1 \text{ is a local minimum}$$

Stage 5: Check for extrema at endpoints

Min possible x: $T(0) = \frac{L^2}{16}$

Max —11— : $T(L) = \frac{L^2}{4\pi} > \frac{L^2}{16}$ $\Rightarrow$

Maximum area is $x = L$, i.e., use all the wire for a circle.