HW07 Solutions

Friday, October 21, 2022 10:31 AM

3.10.15 $f(w) = \cos(\sin^{-1} 2w) = h(g(w))$

Outer function $h(z) = \cos z \qquad h'(z) = -\sin z$

Inner function $g(w) = \sin^{-1} 2w = t$

(*) $\sin t = 2w$

Implicit differentiation to find $\frac{dt}{dw} = t'$

$\frac{d}{dw}$ of (*): $(-\cos t)\, t' = 2 \Rightarrow t' = -\frac{2}{\cos t}$

$\cos t = \sqrt{1-(\sin t)^2} = \sqrt{1-(2w)^2} = \sqrt{1-4w^2}$

$\Rightarrow$

$$g'(w) = -\frac{2}{\sqrt{1-4w^2}}$$

Chain rule: $f'(w) = h'(g(w))\, g'(w) \Rightarrow$

$$h'(g(w)) = -\sin(\sin^{-1} 2w) = -2w$$

$\Rightarrow$

$$f'(w) = \frac{4w}{\sqrt{1-4w^2}}$$

15. $f(w) = \cos(\sin^{-1} 2w)$

16. $f(x) = \sin^{-1}(\ln x)$

17. $f(x) = \sin^{-1}(e^{-2x})$

18. $f(x) = \sin^{-1}(e^{\sin x})$

19. $f(x) = \tan^{-1} 10x$

20. $f(x) = 2x \tan^{-1} x - \ln(1+x^2)$

21. $f(y) = \tan^{-1}(2y^2 - 4)$

22. $g(z) = \tan^{-1}(1/z)$

23. $f(z) = \cot^{-1} \sqrt{z}$

24. $f(x) = \sec^{-1} \sqrt{x}$

25. $f(x) = x^2 + 2x^3 \cot^{-1} x - \ln(1+x^2)$

26. $f(x) = x \cos^{-1} x - \sqrt{1-x^2}$

27. $f(w) = w^2 - \tan^{-1} w^2$

28. $f(t) = \ln(\sin^{-1} t^2)$

29. $f(x) = \cos^{-1}(1/x)$

30. $f(t) = (\cos^{-1} t)^2$

31. $f(u) = \csc^{-1}(2u+1)$

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3.10.19. $f(x) = \tan^{-1} 10x = y$

<u>Method 1</u>: Directly apply rule $\frac{d}{dx} \tan^{-1} x = \frac{d}{dx} \arctan x = \frac{1}{1+x^2}$:

$y(x) = g(h(x))$

Outer function: $g(z) = \tan^{-1} z \,;\; g'(z) = \frac{1}{1+z^2}$

Inner function: $h(x) = 10x \,;\; h'(x) = 10$

$\Rightarrow$

Chain rule: $y'(x) = g'(h(x)) \cdot h'(x) = \frac{10}{1+(10x)^2}$

<u>Method 2</u>: (If you forget $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$)

$y = \tan^{-1} 10x \Rightarrow \tan y = \tan(\tan^{-1} 10x) = 10x$

(since $\tan$, $\tan^{-1}$ are inverses of one another)

Implicit differentiation: $\frac{d}{dx} \tan(y(x)) = \frac{d}{dx}(10x)$

$\frac{d}{dx}(10x) = 10$; $\frac{d}{dx}\tan(y(x)) = \frac{y'}{\cos^2 y} = y'\cdot(1+\tan^2 y) = y'(1+(10x)^2)$

$\Rightarrow \quad y' = \frac{10}{1+(10x)^2}$, as above

Note: $\tan^{-1}x$, $\cos^{-1}x$, $\sin^{-1}x$ are used in this textbook as notations for inverse tan, cos, sin.

This can be confusing since $\tan^2 x$, $\cos^2 x$, $\sin^2 x$ denote the squares of $\tan x$, $\cos x$, $\sin x$, i.e.:

$$\tan^2 x = (\tan x)^2; \quad \cos^2 x = (\cos x)^2; \quad \sin^2 x = (\sin x)^2$$

However, by $\tan^{-1}x$ we do not mean $\frac{1}{\tan x}$.

Rule to apply: $\tan^k x = (\tan x)^k$; $\sin^k x = (\sin x)^k$; $(\cos^k x) = (\cos x)^k$

except for $k=-1$ when $\tan^{-1}x$, $\cos^{-1}x$, $\sin^{-1}x$ mean inverse tan, cos, sin

3.10.21. $f(y) = \tan^{-1}(2y^2-4)$

Chain rule. $f(y) = g(h(y))$

Outer function $g(z) = \tan^{-1}z \qquad g'(z) = \frac{1}{1+z^2}$

Inner function $h(y) = 2y^2-4 \qquad h'(y) = 4y$

$$f'(y) = g'(h(y))\,h'(y) = \frac{4y}{1+(2y^2-4)^2}.$$

3.10.23. $f(z) = \cot^{-1}\sqrt{z}$

Chain rule: $f(z) = h(g(z))$

Inner function: $g(z) = \sqrt{z}$; $g'(z) = \frac{1}{2\sqrt{z}}$

Outer function $h(u) = \cot^{-1}u$; $h'(u) = -\frac{1}{1+u^2}$

$$f'(z) = -\frac{1}{1+(\sqrt{z})^2}\cdot\frac{1}{2\sqrt{z}} = -\frac{1}{2(1+z)\sqrt{z}}$$

$2(1+z)\sqrt{z}$

To find $h'(u)$: $\quad w = \cot^{-1} u \; ; \; \cot w = u$
(if forgotten)

Derivative of inverse rule: $\quad \dfrac{dw}{du} = \dfrac{1}{\frac{du}{dw}}$

$$\frac{du}{dw} = \frac{d}{dw}\cot w = \frac{d}{dw}\frac{\cos w}{\sin w} = \frac{-\sin^2 w - \cos^2 w}{\sin^2 w} = -\frac{1}{\sin^2 w}$$

$\Rightarrow$

$$\frac{dw}{du} = \frac{1}{-\frac{1}{\sin^2 w}} = -\sin^2 w = -\frac{1}{1+\cot^2 w} = -\frac{1}{1+\left[\cot(\cot^{-1} u)\right]^2} = -\frac{1}{1+u^2}.$$

3.10.29. $\quad f(x) = \cos^{-1}\left(\frac{1}{x}\right) = y$

Chain rule $\quad f(x) = g(h(x))$

Inner function $\quad h(x) = \frac{1}{x}\; ; \quad h'(x) = -\frac{1}{x^2}$

Outer function $\quad g(u) = \cos^{-1} u \quad g'(u) = -\frac{1}{\sqrt{1-u^2}}$

$\Rightarrow$

$$f'(x) = g'(h(x))\,h'(x) = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\;\frac{1}{x^2} = \frac{1}{x\sqrt{x^2-1}}$$

To find $g'(u)$ (if $\frac{d}{du}\cos^{-1}u$ was not memorized).

Derivative of inverse: $\quad g = \cos^{-1} u \Rightarrow u = \cos g$

$$\frac{dg}{du} = \frac{1}{\frac{du}{dg}} = \frac{1}{-\sin g} = \frac{1}{-\sqrt{1-\cos^2 g}} = -\frac{1}{\sqrt{1-u^2}}$$

3.11.11.

11. Expanding square The sides of a square increase in length at a rate of $2\,m/s$.

a. At what rate is the area of the square changing when the sides are $10\,m$ long?

b. At what rate is the area of the square changing when the sides are $20\,m$ long?

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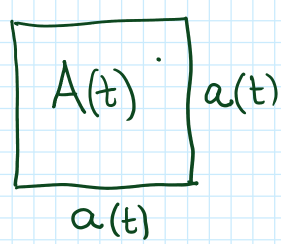

$a(t)\; ; \quad a'(t) = 2\,\frac{m}{s}$

Area of square $A(t) = a^2(t)$

$$A'(t) = 2a(t)\,a'(t)$$

a) When $a(t) = 10\,m$: $A'(t) = 2 \cdot 10 \cdot 2 = 40\ m^2/s$

b) When $a(t) = 20\,m$: $A'(t) = 2 \cdot 20 \cdot 2 = 80\ m^2/s$

3.11.13

13. Expanding isosceles triangle The legs of an isosceles right triangle increase in length at a rate of 2 m/s.

a. At what rate is the area of the triangle changing when the legs are 2 m long?

b. At what rate is the area of the triangle changing when the hypotenuse is 1 m long?

c. At what rate is the length of the hypotenuse changing?

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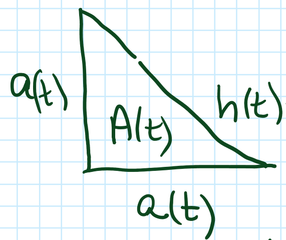

$a'(t) = 2\ m/s$

Area of triangle $A(t) = \frac{1}{2} a^2(t)$; $A'(t) = a(t)\,a'(t)$

a) When $a = 2\,m$: $A'(t) = 2 \cdot 2 = 4\ m^2/s$

b) When $a\sqrt{2} = 1$: $A'(t) = \frac{1}{\sqrt{2}}\, 2 = \sqrt{2}\ m^2/s$

c) $h'(t) = \sqrt{2} \cdot a'(t) = 2\sqrt{2}\ m/s$.

23. Time-lagged flights An airliner passes over an airport at noon traveling 500 mi/hr due west. At 1:00 P.M., another airliner passes over the same airport at the same elevation traveling due north at 550 mi/hr. Assuming both airliners maintain their (equal) elevations, how fast is the distance between them changing at 2:30 P.M.?

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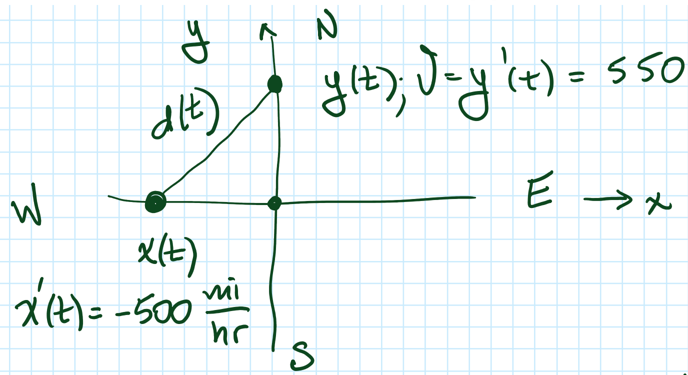
3.11.23. From above.



$y(t)$; $v = y'(t) = 550\ \frac{mi}{hr}$

$u = x'(t) = -500\ \frac{mi}{hr}$

Airliner 1: $x(t) = u \cdot t$ ($t = 0$ at noon, $x(0) = 0$ above airport)

Airliner 2: $y(t) = v \cdot (t-1)$ ($y(1) = 0$, above airport at 1:00 PM)

Distance $d(t) = \sqrt{x^2(t) + y^2(t)}$

Chain rule: $d(t) = f(g(t))$

Outer function: $f(u) = \sqrt{u}$ $f'(u) = \frac{1}{\ldots}$

Chain rule: $d(t) = f(g(t))$

Outer function $f(u) = \sqrt{u}$ $\quad f'(u) = \frac{1}{2\sqrt{u}}$

Inner function $g(t) = x^2(t) + y^2(t)$

$g'(t) = 2x(t)x'(t) + 2y(t)y'(t)$ (Chain rule)

$$\Rightarrow d'(t) = f'(g(t))g'(t) = \frac{x \cdot x' + y \cdot y'}{\sqrt{x^2+y^2}}$$

At 2:30 PM: $x(2.5) = u \cdot (2.5) = (-500) \cdot 2.5 = -1250$ mi

$y(2.5) = v \cdot (2.5-1) = (550) \cdot 1.5 = 825$ mi

$$d'(2.5) = \frac{(-1250)(-500) + (825)(550)}{\sqrt{(1250)^2 + (825)^2}} \cong 720 \text{ mi/hr}$$

3.11.29 See R07

3.11.35.

35. Growing sandpile Sand falls from an overhead bin and accumulates in a conical pile with a radius that is always three times its height. Suppose the height of the pile increases at a rate of 2 cm/s when the pile is 12 cm high. At what rate is the sand leaving the bin at that instant?

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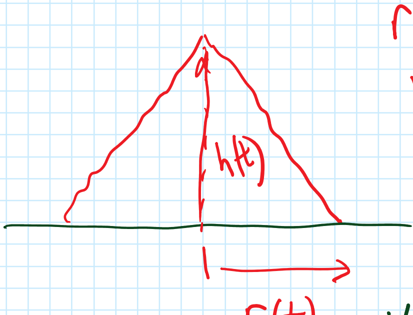

$r(t) = 3h(t)$

$h'(t) = 2 \frac{cm}{s}$ when $h(t) = 12$ cm

"At what rate is sand leaving" =

= At what rate is volume of pile increasing

Volume $V(t) = \frac{\pi}{3} r^2(t) h(t)$

$V'(t) = \frac{\pi}{3}\left[2rr'h + r^2h'\right]$ (product rule)

$r(t) = 3h(t)$; $r'(t) = 3h'(t)$

When $h' = 2 \frac{cm}{s} \Rightarrow r' = 6 \frac{cm}{s}$

$h = 12 \text{ cm} \rightarrow r = 36 \text{ cm}$

$$V'(t) = \frac{\pi}{3}\left[2 \cdot 36 \cdot 6 \cdot 12 + 36^2 \cdot 2\right]$$

$$= 24\pi\,(72 + 36) = 24 \cdot 108 \cdot \pi.$$