HW08 solution

Find critical points (where $f'(c)=0$ or derivative does not exist)

25. $f(x)=\frac{x^3}{3}-9x$

26. $f(x)=\frac{x^4}{4}-\frac{x^3}{3}-3x^2+10$

27. $f(x)=3x^3+\frac{3x^2}{2}-2x$

28. $f(x)=\frac{4x^5}{5}-3x^3+5$

29. $f(x)=x^3-4a^2x$

30. $f(x)=x-5\tan^{-1}x$

31. $f(t)=\frac{t}{t^2+1}$

32. $f(x)=12x^5-20x^3$

33. $f(x)=\frac{e^x+e^{-x}}{2}$

34. $f(x)=\sin x\cos x$

35. $f(x)=\frac{1}{x}+\ln x$

36. $f(t)=t^2-2\ln(t^2+1)$

37. $f(x)=x^2\sqrt{x+5}$

38. $f(x)=(\sin^{-1}x)(\cos^{-1}x)$

39. $f(x)=x\sqrt{x-a}$

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Friday, October 21, 2022 10:31 AM

4.1.25 $f(x)=\frac{x^3}{3}-9x$

f is a polynomial, hence derivative exists on $(-\infty,\infty)$

$f'(x)=x^2-9=0 \Rightarrow x_{1,2}=\pm 3$ are the critical points

4.1.27 $f(x)=3x^3+\frac{3}{2}x^2-2x$

f is polynomial, f' exists on $\mathbb{R}$

$f'(x)=9x^2+3x-2=(3x+2)(3x-1)$

$\Rightarrow$ Critical points are $x_1=-\frac{2}{3}$ $x_2=\frac{1}{3}$

4.1.31 $f(t)=\frac{t}{t^2+1}$

$$f'(t)=\frac{t^2+1-t(2t)}{(t^2+1)^2}=\frac{1-t^2}{(t^2+1)^2}$$

f' exists for $t\in\mathbb{R}$

Critical points $f'(t)=0 \Rightarrow t_{1,2}=\pm 1$

4.1.39 $f(x)=x\sqrt{x-a}$ $f:[a,\infty)\to\mathbb{R}$

$$f'(x)=\sqrt{x-a}+\frac{x}{2\sqrt{x-a}} \qquad f':(a,\infty)\to\mathbb{R}$$

$x=a$ is a critical point (f' does not exist)

$$f'(x)=0 \Rightarrow \sqrt{x-a}+\frac{x}{2\sqrt{x-a}}=0 \Rightarrow 2(x-a)=-x \Rightarrow x=\frac{2a}{3}$$

If $a>0$ then $\frac{2a}{3}<a$, and $x=\frac{2a}{3}$ is not a critical point since it is outside the function domain

If $a<0$ then $\frac{2a}{3}>a$, and $x=\frac{2a}{3}$ is a critical point.

4.1.72.

72. Minimum-surface-area box All boxes with a square base and a volume of 50 ft^3 have a surface area given by $S(x)=2x^2+\frac{200}{x}$, where x is the length of the sides of the base. Find the absolute minimum of the surface area function on the interval $(0,\infty)$. What are the dimensions of the box with minimum surface area?

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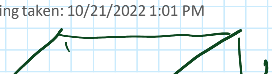

Surface area $S=4hx+2x^2$ $\Big\}\Rightarrow$

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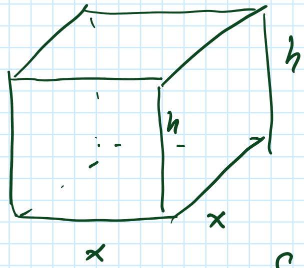

Surface area $S = 4hx + 2x^2$

Volume $V = hx^2 = 50 \Rightarrow h = \dfrac{50}{x^2}$ $\Bigg\} \Rightarrow$

$$S(x) = \frac{200}{x} + 2x^2$$

$$S'(x) = 4x - \frac{200}{x^2}; \quad S'(x) = 0 \Rightarrow x^3 = 50 \Rightarrow x_1 = \sqrt[3]{50}$$

$$S''(x) = 4 + \frac{400}{x^3} > 0 \text{ for } x > 0 \Rightarrow x_1 = \sqrt[3]{50} \text{ is a minimum}$$

Dimensions of box with minimum surface area are

Side $= x_1 = \sqrt[3]{50}$; height $= h_1 = \dfrac{50}{x_1^2} = \sqrt[3]{50}$ (a cube).

4.1.73

73. Trajectory high point A stone is launched vertically upward from a cliff 192 ft above the ground at a speed of 64 ft/s. Its height above the ground t seconds after the launch is given by $s = -16t^2 + 64t + 192$, for $0 \le t \le 6$. When does the stone reach its maximum height?

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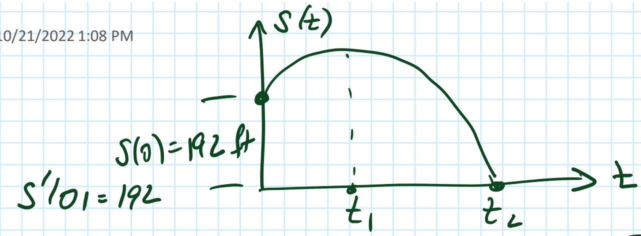

$$s(t) = -16t^2 + 64t + 192; \quad s: [0, 6] \to \mathbb{R}$$

$$s'(t) = -32t + 64; \quad s'(t) = 0 \Rightarrow t_1 = 2$$

$$s''(t) = -32 < 0 \Rightarrow s(t_1) \text{ is a local maximum.}$$

21–32. Mean Value Theorem *Consider the following functions on the given interval $[a, b]$.*

- **a.** *Determine whether the Mean Value Theorem applies to the following functions on the given interval $[a, b]$.*
- **b.** *If so, find the point(s) that are guaranteed to exist by the Mean Value Theorem.*

21. $f(x) = 7 - x^2$; $[-1, 2]$

22. $f(x) = x^3 - 2x^2$; $[0, 1]$

23. $f(x) = \begin{cases} -2x & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases}$; $[-1, 1]$

24. $f(x) = \dfrac{1}{(x-1)^2}$; $[0, 2]$

25. $f(x) = e^x$; $[0, 1]$

26. $f(x) = \ln 2x$; $[1, e]$

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4.2.25. $f(x) = e^x$ is continuous and differentiable, MVT applies

There must exist $0 < c < 1$ such that $f'(c) = \dfrac{f(1) - f(0)}{1 - 0} \Rightarrow$

$$e^c = \frac{e - 1}{1 - 0} = e - 1 \Rightarrow c = \ln(e - 1).$$

$$e^c = \frac{e-1}{1-0} = e-1 \Rightarrow c = \ln(e-1).$$

4.2.26. $f(x) = \ln 2x \qquad f: [1, e]$

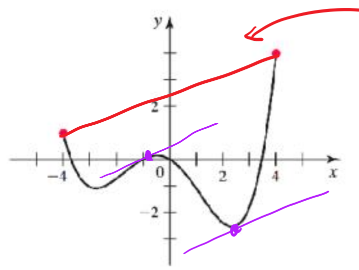
f is continuous on $[1,e]$ and differentiable on $(1,e)$

MVT applies.

$\exists\, c \in (1,e)$ such that $\displaystyle \frac{1}{c} = \frac{\ln 2e - \ln 1}{e-1} = \frac{1+\ln 2}{e-1}$

4.2.39.

39. Mean Value Theorem and graphs By visual inspection, locate all points on the interval $(-4, 4)$ at which the slope of the tangent line equals the average rate of change of the function on the interval $[-4, 4]$.



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4.2.45.

45. Running pace Explain why if a runner completes a 6.2-mi (10-km) race in 32 min, then he must have been running at exactly 11 mi/hr at least twice in the race. Assume the runner's speed at the finish line is zero.

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Average speed $\displaystyle V = \frac{6.2}{(32/60)} = \frac{62\cdot 6}{32} = \frac{31\cdot 3}{8} = \frac{93}{8} \cong 11.6$ mi/hr

Let $v(t)$ denote instantaneous speed, continuous, differentiable

We know $v(0)=0 \qquad v(32/60)=0$

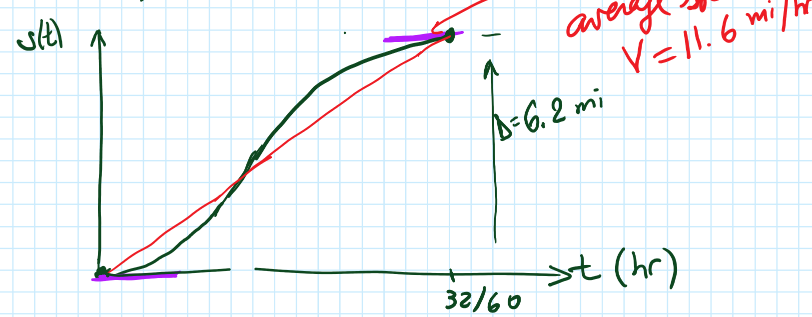
Let $s(t)$ denote distance $\qquad s(0)=0 \qquad s(t)=6.2$

$s'(t) = v(t)$

$s'(0) = 0$

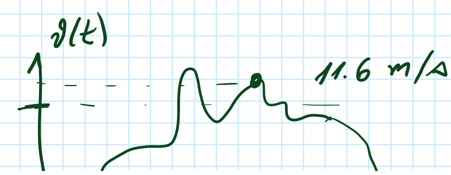
$s'(32/60) = 0$

Slopes of tangent lines at $t=0,\ t=\frac{32}{60}$



MVT: $\exists\, t_1$ such that $\displaystyle s'(t_1) = \frac{s(\frac{32}{60}) - s(0)}{\frac{32}{60}} = V = 11.6$ mi/hr

Apply Intermediate value theorem twice on $[0, t_1]$ & $[t_1, \frac{32}{60}]$



Apply [illegible]

on $[0, t_1]$ & $[t_1, \frac{32}{60}]$

$\Rightarrow$ there must exist points at which

velocity is $v(t) = 11$ $\quad 0 < 11 < 11.6$.