HW09 solution

Monday, October 31, 2022 8:21 AM

15. Is it possible for a function to satisfy $f(x) > 0$, $f'(x) > 0$, and $f''(x) < 0$ on an interval? Explain.

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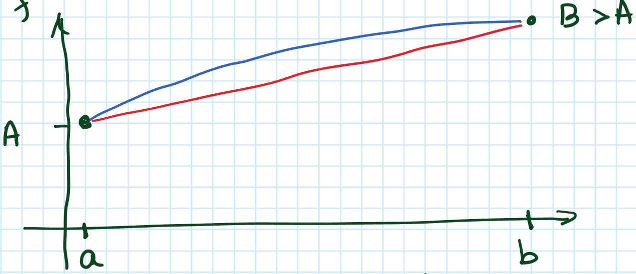
Choose an arbitrary interval (a,b)

Put information into table on

x	a							b
f	A +	+	+	+	+	+		B
f'		+	+	+	+	+	+	
f''		−	−	−	−	−	−	

$$A = \lim_{x\to a} f(x), \quad B = \lim_{x\to b} f(x)$$

Transcribe information into a plot



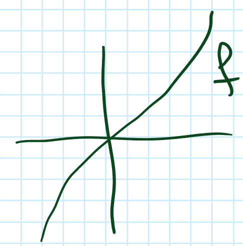
— This function has zero second derivative (does not satisfy)

— This function satisfies all conditions.

$\Rightarrow$ Yes, it is possible

4.3.22. $f(x) = x^3 + 4x \Rightarrow f'(x) = 3x^2 + 4$

f' always positive means f always increasing



25, 48, 50

22. $f(x) = x^3 + 4x$

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27. $f(x) = -\frac{x^4}{4} + x^3 - x^2$

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4.3 27. (Extra solution) $f(x) = -\frac{x^4}{4} + x^3 - x^2 \Rightarrow$

$$f'(x) = -x^3 + 3x^2 - 2x = (-x)(x^2 - 3x + 2)$$

$$f'(x) = (-x)(x-2)(x-1)$$

f' changes sign at simple roots solutions of $f'(x) = 0$.

Roots are: $x_1 = 0$, $x_2 = 2$, $x_3 = 1$

Table:

x	$-\infty$		0			1			2			∞
f	$-\infty$	↗	0	↘		$-\frac{1}{4}$	↗		0	↘		$-\infty$
f'	+	+	0	−	−	0	+	+	0	−	−	−

f Increasing on intervals $(-\infty, 0)$ $(-\frac{1}{4}, 0)$

f Decreasing —"— $(0, 1)$ $(0, \infty)$

31. $f(x) = -2\cos x - x$ on $[0, 2\pi]$

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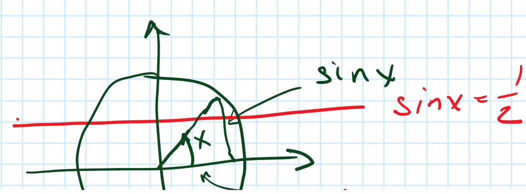
4.3.31 $f'(x) = 2\sin x - 1$

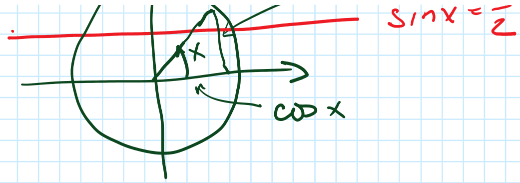
Table:

x	0		$\frac{\pi}{6}$		$\frac{5\pi}{6}$		2π
f	-2	↘		↗		↘	-2π
f'	-1	−	0	+	0	−	-1

Roots: $f'(x) = 0 \Rightarrow \sin x = \frac{1}{2}$. Unit circle



Roots: $f'(x)=0 \Rightarrow \sin x = \frac{1}{2}$. Unit circle



$\Rightarrow x_1 = \frac{\pi}{6} \quad x_2 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

f' changes sign at roots

f Increasing intervals: $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

f Decreasing intervals: $\left(0, \frac{\pi}{6}\right), \left(\frac{5\pi}{6}, 2\pi\right)$

4.3.39.

(Extra Solution) **39.** $f(x) = -12x^5 + 75x^4 - 80x^3$

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$f'(x) = -60x^4 + 300x^3 - 240x^2$

$f'(x) = (-60x^2)(x^2 - 5x + 4)$

$f'(x) = (-60x^2)(x-4)(x-1)$

Roots of f'
$x_{1,2} = 0$ (twice)
$x_3 = 1$
$x_4 = 4$

Table:

x	$-\infty$		0		1		4		∞
f	∞	↘		↘		↗		↘	$-\infty$
f'		$- -$	0	$- - - -$	0	$+ +$	0	$- -$	

Observation: When a double root is encountered, the function does not necessarily change sign. Check signs on both sides of root.

f increasing on interval $(1, 4)$

f decreasing on intervals $(-\infty, 0)$ $(0, 1)$ $(4, \infty)$

4.3.72.

(Extra Solution) **72.** $p(x) = x^4 e^x + x$

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$p'(x) = 4x^3e^x + x^4e^x + 1 = x^3e^x(x+4) + 1$

$p'(x) = e^x(x^4 + 4x^3) + 1$

$p''(x) = e^x(4x^3 + 12x^2) + e^x(x^4 + 4x^3) \Leftrightarrow$

$p''(x) = x^2e^x(x^2 + 8x + 12) = x^2e^x(x+6)(x+2)$

Roots of $p''(x) = 0$ $\quad x_{1,2} = 0$ (double root)
$x_3 = -2$ simple root
$x_4 = -6$ simple root

Table

x	$-\infty$		-6		-2		0		∞
f	$-\infty$	∪		∪		∩		∪	∞
f'	1								∞
f''	0	$+ +$	0	$+ +$	0	$- -$	0	$+ + +$	∞

f concave up "holds water" intervals: $(-\infty, -6)$ $(-6, -2)$

f —"— down "does not hold water" —"—: $(-2, 0)$

At roots of f'' f is neither concave up nor down. It's an inflection point.

[illegible]

f — down does not hold water — : $(-2, 0)$ is an inflection point.

4.3.73. (Extra Solution)

73. $f(x) = \sqrt{x} \ln x$

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$$f'(x) = \frac{1}{2\sqrt{x}} \ln x + \sqrt{x}\,\frac{1}{x} = \frac{1}{\sqrt{x}}\left(\frac{1}{2}\ln x + 1\right)$$

$$f''(x) = -\frac{1}{2x^{3/2}}\left(\frac{1}{2}\ln x + 1\right) + \frac{1}{\sqrt{x}}\left(\frac{1}{2x}\right) \Rightarrow$$

$$f''(x) = \frac{1}{2x^{3/2}}\left(1 - \frac{1}{2}\ln x - 1\right) = -\frac{1}{4x^{3/2}}\ln x$$

f defined for $x > 0$, $f: (0, \infty)$

Roots of f'': $f''(x) = 0 \Rightarrow -\frac{1}{4x^{3/2}}\ln x = 0$ has root $x_1 = 1$

Table:

x	0		1		∞
		∪		∩	
f''		+ + +	0	− − − − −	

f concave up interval $(0, 1)$

f concave down interval $(1, \infty)$

(25, 48, 50)

4.3.25.

25. $f(x) = 12 + x - x^2$

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$$f'(x) = 1 - 2x$$

Roots $f'(x) = 0 \Rightarrow x_1 = \frac{1}{2}$

Table

x	$-\infty$		$\frac{1}{2}$		∞
f	$-\infty$	↗		↘	$-\infty$
f'		+ + +	0	− − −	

f increasing interval $(-\infty, \frac{1}{2})$

f decreasing interval $(\frac{1}{2}, \infty)$

4.3.48 .

48. $f(x) = 2x^3 + 3x^2 - 12x + 1$ on $[-2, 4]$

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$$f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2)$$

$$f'(x) = 6(x-1)(x+2)$$

Roots of $f'(x) = 0$ $x_1 = 1$; $x_2 = -2$

Table

x	-2		1		4
f	21	↘ ↘	-6	↗ ↗	129
f'	0	− − −	0	+ + +	

"Absolute" or "Global" maximum is at $x_1 = 4 \Rightarrow f(4) = 129$

$x_2 = 1$ is both a local & global minimum.

4.3.50

$$f'(x) = 10x^4 - 20x^3 - 30x^2$$

$x_2 = 1$ is both a local ...

4.3.50.

50. $f(x) = 2x^5 - 5x^4 - 10x^3 + 4$ on $[-2, 4]$

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$$f'(x) = 10x^4 - 20x^3 - 30x^2$$
$$f'(x) = 10x^2(x^2 - 2x - 3) \Rightarrow$$
$$f'(x) = 10x^2(x-3)(x+1)$$

Roots: $x_{1,2} = 0$ (double root) f' does not necessarily change sign

$x_3 = 3$ (simple root) f' changes sign

$x_4 = -1$ — " — f' — " —

Table:

x	-2		-1		0		3		4
f	-60	↗	7	↘	4	↘	-185	↗	132
f'		$+ +$	0	$- -$	0	$- -$	0	$+ + +$	

Absolute/Global maximum at $x=4$ endpoint $f(4) = 132$

f has a local maximum at $x = -1$

Absolute/Global minimum at $x = 3$, also a local minimum

$x = 4$ is an inflection point.

4.4.7

7. $f' < 0$ and $f'' < 0$, for $x < 3$
$f' < 0$ and $f'' > 0$, for $x > 3$

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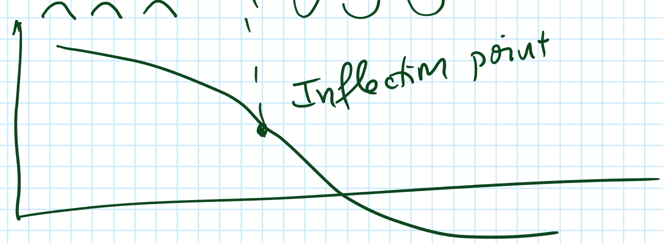
Table:

(7, 8, 17)

x		3	
f	↘		↘
f'	$- - -$		$- - -$
f''	$- - -$ ∩ ∩ ∩		$+ + +$ ∪ ∪ ∪



4.4.8.

8. $f' < 0$ and $f'' < 0$, for $x < -1$
$f' < 0$ and $f'' > 0$, for $-1 < x < 2$
$f' > 0$ and $f'' > 0$, for $2 < x < 8$
$f' > 0$ and $f'' < 0$, for $8 < x < 10$
$f' > 0$ and $f'' > 0$, for $x > 10$

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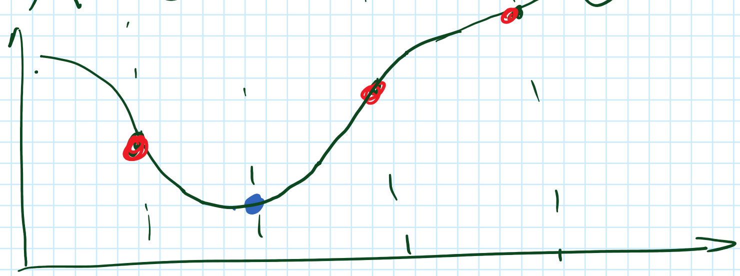
Table

x		-1		2		8		10	
f	↘		↘		↗		↗		↗
f'	$- -$		$- -$		$+ + +$		$+ + +$		$+ + +$
f''	$- -$ ∩		$+ +$ ∪		$+ + +$ ∪		$- - -$ ∩		$+ + +$ ∪

Plot



● inflection pts

• local extrema

4.4.17.

17. $f(x) = x^3 - 6x^2 + 9x$

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$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$

Roots $f'(x) = 0 \Rightarrow x_1 = 1, \quad x_2 = 3$ (simple roots)

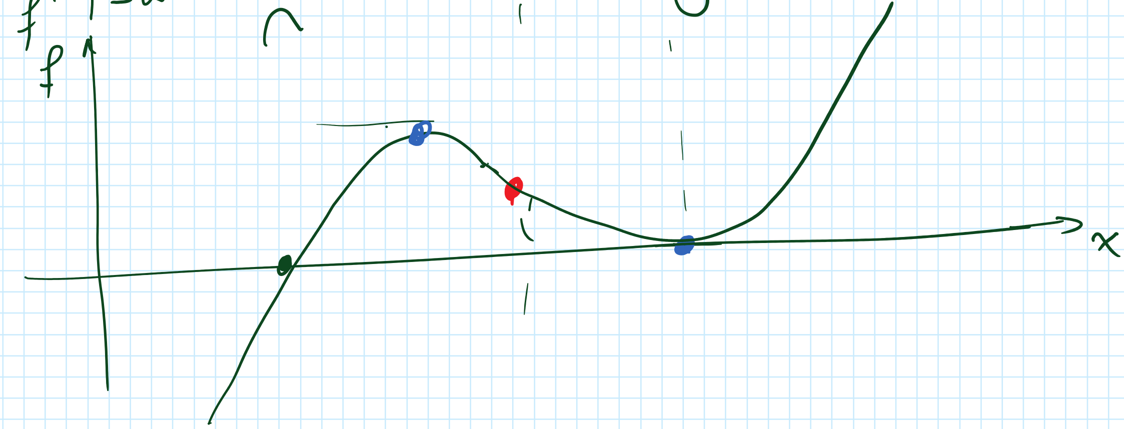
$f''(x) = 6x - 12 = 6(x-2)$

Roots $f''(x) = 0 \Rightarrow x_3 = 2$ (simple roots)

Table.

x	$-\infty$		0		1		2		3		∞
f	$-\infty$	↗	0	↗	4	↘		↘	0	↗ ↗	∞
f'	∞	+		+	0	−		−	0	+ +	∞
f''	$-\infty$			−		−	0	+		+ +	∞
		∩						∪			

Graph aligned with table



4.4.19

19. $f(x) = x^4 - 6x^2$

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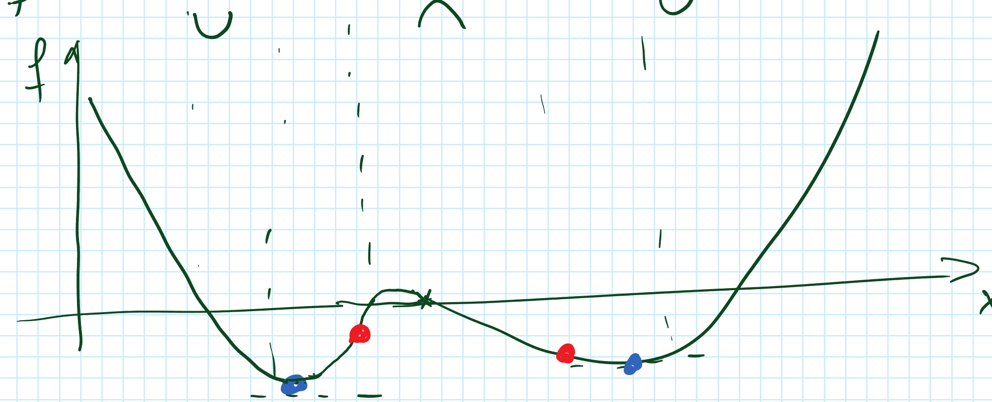
$f'(x) = 4x^3 - 12x = 4x(x^2 - 3)$ roots: $x_1 = -\sqrt{3}, x_2 = 0, x_3 = \sqrt{3}$ (1,2,3)

$f''(x) = 12x^2 - 12 = 12(x^2 - 1)$ roots $x_4 = -1 \quad x_5 = 1$

Table:

x	$-\infty$	$-\sqrt{6}$	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	$\sqrt{6}$	∞
f	∞ ↘	0 ↘		↗	0	↘		0 ↗	∞
f'	$-\infty$	− −	0	+ +	0	− −	0	+ + +	∞
f''	∞	+ +		0	− −	0	+ + +		∞
		∪			∩		∪		

Graph aligned with table



4.4.23.

23. $f(x) = x^3 - 6x^2 - 135x$

$f'(x) = 3x^2 - 12x - 135 = 3(x^2 - 4x - 45)$

$f'(x) = 3(x-9)(x+5) \Rightarrow x_1 = -5 \quad x_2 = 9$

23. $f(x) = x^3 - 6x^2 - 135x$

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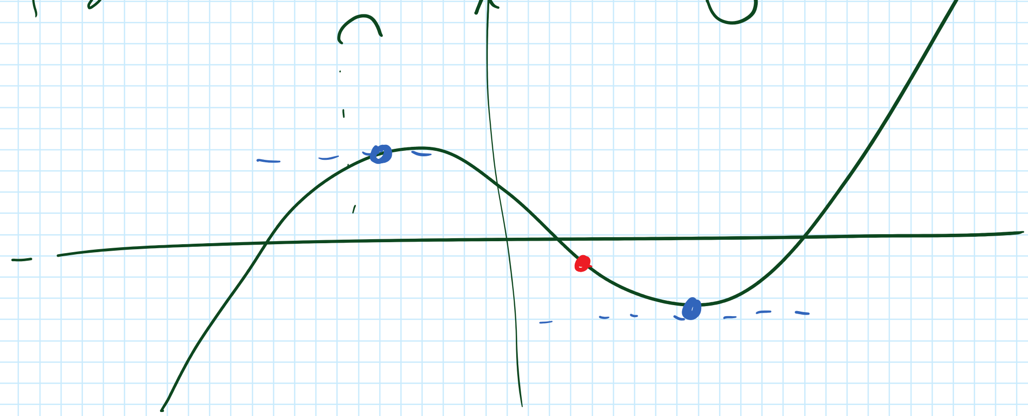
$f'(x) = 3(x-9)(x+5) \Rightarrow x_1 = -5 \quad x_2 = 9$

$f''(x) = 6x - 12 = 6(x-2)$

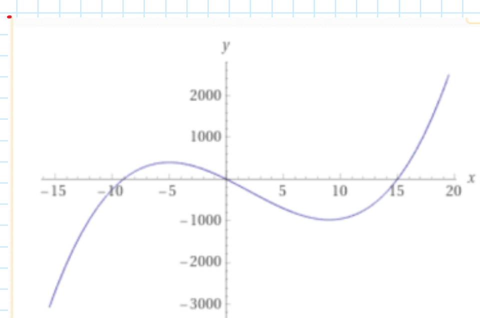
$f''(x) = 0 \Rightarrow x_3 = 2$

Table

x	$-\infty$		-5		0		2		9		∞
f	$-\infty$	$\nearrow$		$\searrow$	0	$\searrow$				$\nearrow$	∞
f'	∞	$+ \; +$	0	$- \; -$					0	$+ \; +$	∞
f''	$-\infty$	$- \; -$	$-$	$-$			0	$+$	$+$	$+$	∞
			$\cap$					$\cup$			



Compare with computer-generated plot (qualitatively the same ✓).



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