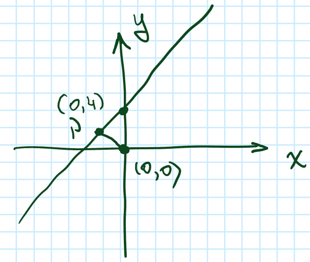
HW 10 Solutions

Monday, November 7, 2022 10:51 AM

4.5.22.



22. Closest point on a line What point on the line $y = 3x + 4$ is closest to the origin?

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Squared Distance from (0,0) of point (x,y) is

$$D_2(x,y) = x^2 + y^2$$

condition for (x,y) to be on line $y = 3x + 4$ $\Big\} \Rightarrow \quad d_2(x) = x^2 + (3x+4)^2$

Distance is minimal when squared distance is minimal. Seek critical pts.

$$d_2'(x) = 2x + 2\cdot(3x+4)\cdot(3) = 20x + 24 = 0 \Rightarrow x = -\frac{6}{5}$$

$$d_2''(x) = 20 > 0 \Rightarrow \text{critical pt. is a minimum}$$

Solution: $P\left(-\frac{6}{5}, 3\left(-\frac{6}{5}\right) + 4\right) = P\left(-\frac{6}{5}, \frac{2}{5}\right)$.

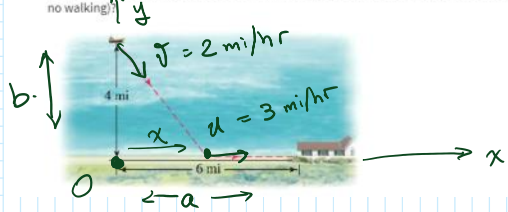
4.5.27. Notation: $b = 4$ mi, $a = 6$ mi; $v = 2$ mi/hr; $u = 3$ mi/hr

$$T(x) = \text{total travel time} = T_{row} + T_{walk} = \frac{(x^2+b^2)^{1/2}}{v} + \frac{a-x}{u}$$

27. Walking and rowing A boat on the ocean is 4 mi from the nearest point on a straight shoreline; that point is 6 mi from a restaurant on the shore (see figure). A woman plans to row the boat straight to a point on the shore and then walk along the shore to the restaurant.

a. If she walks at 3 mi/hr and rows at 2 mi/hr, at which point on the shore should she land to minimize the total travel time?

b. If she walks at 3 mi/hr, what is the minimum speed at which she must row so that the quickest way to the restaurant is to row directly (with no walking)?



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a) Critical pt.: $T'(x) = 0 \Rightarrow \frac{x}{v(x^2+b^2)^{1/2}} - \frac{1}{u} = 0 \Rightarrow$

$$\frac{u}{v}x = (x^2+b^2)^{1/2} \Rightarrow \left(\frac{u}{v}\right)^2 x^2 = x^2 + b^2 \Rightarrow$$

$$\left[\left(\frac{u}{v}\right)^2 - 1\right]x^2 = b^2 \Rightarrow x_1 = \frac{b}{\left[\left(\frac{u}{v}\right)^2 - 1\right]^{1/2}} = \frac{b\cdot v}{\sqrt{u^2 - v^2}} = \frac{4\cdot 2}{\sqrt{5}} = \frac{8}{\sqrt{5}} \cong 3.6$$

$$0 < x_1 < 6$$

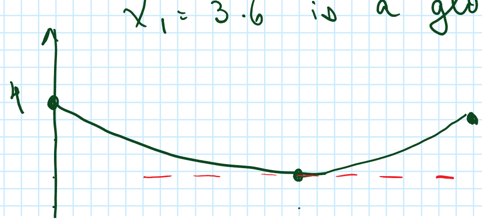
$$T(x_1) = \frac{\left(\frac{64}{5} + 16\right)^{1/2}}{2} + \frac{6 - \frac{8}{\sqrt{5}}}{3} = \frac{12}{10} + \frac{6 - \frac{8}{\sqrt{5}}}{3} \cong 2$$

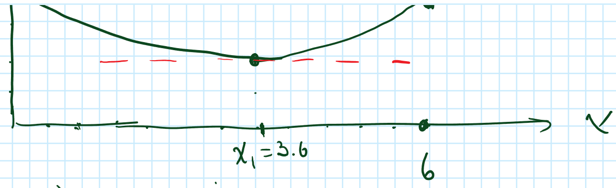
Check endpoints $T(0) = \frac{b}{v} + \frac{a}{u} = \frac{4}{2} + \frac{6}{3} = 4 > T(x_1)$

$$T(a) = \frac{(a^2+b^2)^{1/2}}{v} = \frac{(6^2+4^2)^{1/2}}{2} = \frac{(36+16)^{1/2}}{2} = \frac{\sqrt{52}}{2} \cong 3.6 > T(x_1)$$

$\Rightarrow$ $x_1 = 3.6$ is a global ("absolute") minimum.

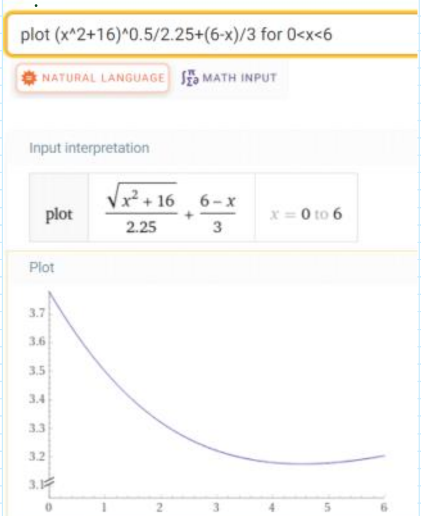
Plot of T(x) by hand & computer





b) As rowing speed increases, the local minimum moves to the right

For $v = 2.25 > 2$:



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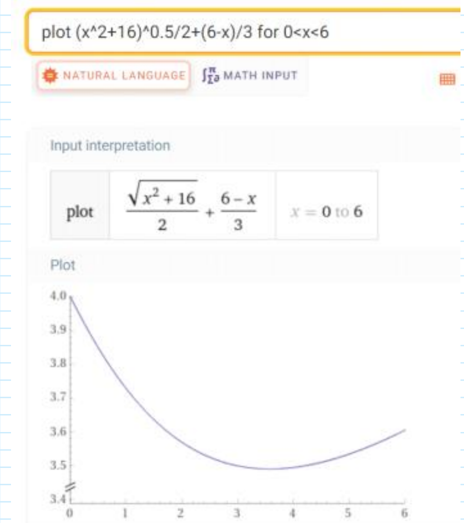
For some v the local minimum occurs at $x = 6$

$$T'(6) = \frac{36}{v(36+16)^{1/2}} - \frac{1}{3} = 0$$

$$\Rightarrow v = \frac{1}{3 \cdot 36} \cdot (36+16)^{1/2}.$$



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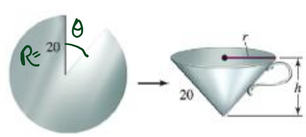
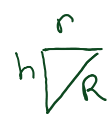
4.5.33. Notation: $R = 20$; Angle of cut sector $= \theta$.

33. Maximum-volume cone A cone is constructed by cutting a sector from a circular sheet of metal with radius 20. The cut sheet is then folded up and welded (see figure). Find the radius and height of the cone with maximum volume that can be formed in this way.





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Formulas: Area of metal sheet

$$A = \left(\pi - \frac{\theta}{2}\right) R^2$$

Lateral area of cone: $S = \pi r \sqrt{r^2 + h^2}$

Volume of cone: $V = \frac{\pi}{3} h r^2$

Equal areas $\Rightarrow$ $A = S$ $\Rightarrow$ $\left(\pi - \frac{\theta}{2}\right) R^2 = \pi r \sqrt{r^2 + h^2}$ $\Bigg| \Rightarrow$

Cone cross section : $R^2 = r^2 + h^2$

Pythagora's theorem

$$\left(\pi - \frac{\theta}{2}\right) R^2 = \pi r R \quad (1)$$

$$V = \frac{\pi}{3} h (R^2 - h^2)$$

R fixed, Max. volume $\Rightarrow$ $V'(h) = 0 \Rightarrow$ $\frac{\pi R^2}{3} - \pi h^2 = 0 \Rightarrow$

$h = \frac{R}{\sqrt{3}}$ $\Rightarrow$ $r^2 = R^2 - \frac{R^2}{3} \Rightarrow r = \sqrt{\frac{2}{3}}\, R$ (2)

R fixed, Max. volume $\Rightarrow$ $V'(h) = 0 \Rightarrow$ ... 3

$h = \frac{R}{\sqrt{3}} \Rightarrow r^2 = R^2 - \frac{R^2}{3} \Rightarrow r = \sqrt{\frac{2}{3}}\, R \quad (2)$

Replace (2) into (1): $\left(\pi - \frac{\theta}{2}\right) R^2 = \pi \sqrt{\frac{2}{3}}\, R^2 \Rightarrow$

$\pi - \frac{\theta}{2} = \pi\sqrt{\frac{2}{3}} \Rightarrow \theta = 2\pi\left(1 - \sqrt{\frac{2}{3}}\right) = 0.367$

4.5.40 Notation:

40. Folded boxes

a. Squares with sides of length x are cut out of each corner of a rectangular piece of cardboard measuring 5 ft by 8 ft. The resulting piece of cardboard is then folded into a box without a lid. Find the volume of the largest box that can be formed in this way.

b. Squares with sides of length x are cut out of each corner of a square piece of cardboard with sides of length l. Find the volume of the largest open box that can be formed in this way.

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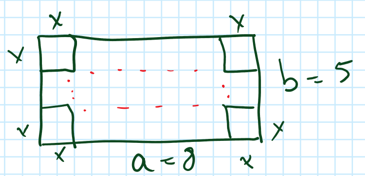


a) ---- folding lines

Volume: $V = A \cdot x$

x = box height

A = box bottom area = $(a-2x)(b-2x)$

$V(x) = (a-2x)(b-2x)x = abx - 2(a+b)x^2 + 4x^3$

$V'(x) = 12x^2 - 4(a+b)x + ab = 0 \Rightarrow$

$x^2 - \frac{1}{3}(a+b)x + \frac{ab}{12} = 0$

$\left.\begin{array}{l} a = 8 \\ b = 5 \end{array}\right\} \Rightarrow x^2 - \frac{13}{3}x + \frac{10}{3} = 0 \Rightarrow 3x^2 - 13x + 10 = 0$

$(3x - 10)(x - 1) = 0 \Rightarrow x_1 = 1$

$x_2 = \frac{10}{3}$

$V''(x) = 12\left(2x - \frac{1}{3}(a+b)x\right) = 12\left(2x - \frac{13}{3}x\right)$

$V''(x_1) = V''(1) = 12\left(2 - \frac{13}{3}\right) < 0 \Rightarrow x_1$ is a maximum

$V''(x_2) = 12\left(\frac{20}{3} - \frac{13}{3}\right) > 0 \Rightarrow x_2$ is a minimum

$\Rightarrow$ Largest volume obtained for $x_1 = 1$

$V(x_1) = V(1) = (8-2)(5-2)\cdot 1 = 6 \cdot 3 = 18$

b) Same as above with $a = b = l$ (utility of general notation)

$V(x) = (l - 2x)^2 x$

$V'(x) = 12x^2 - 8lx + l^2 = 0 \Rightarrow$

$(6x - l)(2x - l) = 0 \Rightarrow x_1 = \frac{l}{6} \quad x_2 = \frac{l}{2}$

$V''(x) = 24x - 8l$

$V''(x_1) = 24\frac{l}{6} - 8l = -4l < 0 \Rightarrow$ maximum

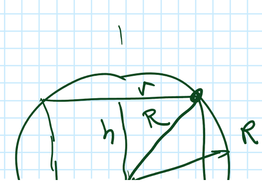
$V''(x_2) = 24\frac{l}{2} - 8l = 4l > 0 \Rightarrow$ minimum

Max Volume for $x_2 = \frac{l}{6} \Rightarrow V(x) = \left(\frac{2l}{3}\right)^2 \frac{l}{6} = \frac{2l^3}{27}$.

4.5.45. Notation & sketch R sphere radius

r cylinder radius

h — " — height



45. Maximum-volume cylinder in a sphere Find the dimensions of the right circular cylinder of maximum volume that can be placed inside of a

4.5.45. Notation & sketch

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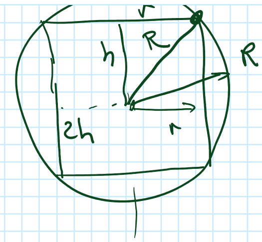

45. Maximum-volume cylinder in a sphere Find the dimensions of the right circular cylinder of maximum volume that can be placed inside of a sphere of radius R.

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Pythagorean th. $R^2 = r^2 + h^2$

Cyl. volume: $V = \pi r^2 2h$

$\Rightarrow V(h) = \pi (R^2 - h^2) 2h$

$V(h) = 2\pi h (R^2 - h^2)$

Critical pts: $V'(h) = 2\pi (R^2 - 3h^2) = 0 \Rightarrow h_1 = \frac{R}{\sqrt{3}}$

$V''(h) = -12\pi h < 0 \Rightarrow$ maximum

$\rightarrow r_1^2 = R^2 - h_1^2 = \frac{2}{3} R^2 \Rightarrow r_1 = \sqrt{\frac{2}{3}} R.$

21. $\lim_{x\to 1} \frac{\ln x}{4x - x^2 - 3}$

22. $\lim_{x\to 0} \frac{e^x - 1}{x^2 + 3x}$

23. $\lim_{x\to \infty} \frac{3x^4 - x^2}{6x^4 + 12}$

24. $\lim_{x\to \infty} \frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4}$

25. $\lim_{x\to e} \frac{\ln x - 1}{x - e}$

26. $\lim_{x\to 1} \frac{4\tan^{-1} x - \pi}{x - 1}$

27. $\lim_{x\to 0^+} \frac{1 - \ln x}{1 + \ln x}$

28. $\lim_{x\to 0^+} \frac{x - 3\sqrt{x}}{x - \sqrt{x}}$

29. $\lim_{x\to 0} \frac{3\sin 4x}{5x}$

30. $\lim_{x\to 2\pi} \frac{x\sin x + x^2 - 4\pi^2}{x - 2\pi}$

31. $\lim_{u\to \pi/4} \frac{\tan u - \cot u}{u - \pi/4}$

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4.7.21. $L = \lim_{x\to 1} \frac{\ln x}{4x - x^2 - 3}$ "$\frac{0}{0}$" indeterm.

l'Hôpital $\Rightarrow L = \lim_{x\to 1} \frac{\frac{1}{x}}{4 - 2x} = \frac{1}{2}$

Check:

lim ln(x)/(4x-x^2-3) as x->1

NATURAL LANGUAGE MATH

Limit

$\lim_{x\to 1} \frac{\log(x)}{4x - x^2 - 3} = \frac{1}{2}$

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4.7.25 $L = \lim_{x\to e} \frac{\ln x - 1}{x - e}$ "$\frac{0}{0}$" indet. ✓

l'Hôpital: $L = \lim_{x\to e} \frac{\frac{1}{x}}{1} = \frac{1}{e}$

4.7.31 $L = \lim_{u\to \pi/4} \frac{\tan u - \cot u}{u - \pi/4}$ "$\frac{0}{0}$" indet. ✓

l'Hôpital $L = \lim_{u\to \frac{\pi}{4}} \frac{\sec^2 u + \csc^2 u}{4} = 1.$

35. $\lim_{x\to \pi} \frac{\cos x + 1}{(x - \pi)^2}$

36. $\lim_{x\to 0} \frac{e^x - x - 1}{5x^2}$

37. $\lim_{x\to \pi/2^-} \frac{\tan x}{3/(2x - \pi)}$

38. $\lim_{x\to \infty} \frac{e^{3x}}{3e^{3x} + 5}$

39. $\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^4 + 8x^3 + 12x^2}$

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4.7.35 $L = \lim_{x\to \pi} \frac{\cos x + 1}{(x - \pi)^2}$ "$\frac{0}{0}$" indet. ✓

l'Hôpital $L = \lim_{x\to \pi} \frac{-\sin x}{2(x - \pi)}$ "$\frac{0}{0}$" —"— ✓

l'Hôpital $L = \lim_{x\to \pi} \frac{-\cos \pi}{2x} = \frac{1}{2\pi}.$

4.7.39. $L = \lim_{x\to 0} \frac{e^x - \sin x - 1}{x^4 + 8x^3 + 12x^2}$ "$\frac{0}{0}$" indet. ✓

l'Hôpital: $L = \lim_{x\to 0} \frac{e^x - \cos x}{4(x^3 + 3x^2 + 6x)}$ "$\frac{0}{0}$" —"—

$- \parallel -$ $$L = \lim_{x \to 0} \frac{e^x + \sin x}{12(x^2 + 2x + 2)} = \frac{1}{24}.$$

Check:

lim (exp(x)-sin(x)-1)/(x^4+8x^3+12x^2) as x->0

NATURAL LANGUAGE MATH INPUT

Limit

$$\lim_{x \to 0} \frac{\exp(x) - \sin(x) - 1}{x^4 + 8x^3 + 12x^2} = \frac{1}{24}$$

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