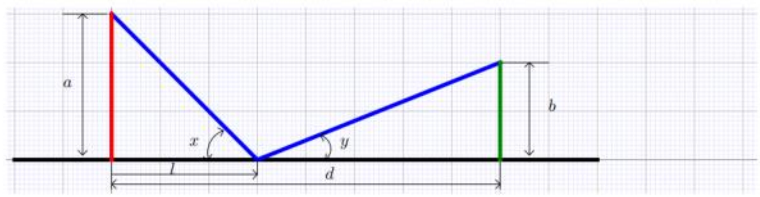
Test 3 Solution

Tuesday, November 15, 2022 10:19 AM

1.

1. A **rope** is stretched from the top of a **pole** of height a to the **ground** at variable distance l from the **pole**, and back up to another **pole** of height b that is distance d away from the first pole. Show that the minimum amount of rope needed occurs when angle x equals angle y.



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Independent variable: ℓ

Dependent variable: $L(\ell)$ rope length

$$L(\ell) = \sqrt{a^2+\ell^2} + \sqrt{b^2+(d-\ell)^2}$$

Min. rope length $\Rightarrow$ $L'(\ell) = 0$

$$L'(\ell) = \frac{\ell}{\sqrt{a^2+\ell^2}} - \frac{d-\ell}{\sqrt{b^2+(d-\ell)^2}}$$

$$\left.\begin{aligned} \cos x &= \frac{\ell}{\sqrt{a^2+\ell^2}} \\ \cos y &= \frac{d-\ell}{\sqrt{b^2+(d-\ell)^2}} \end{aligned}\right\} \Rightarrow$$

$L'(\ell) = 0 \Rightarrow \cos x = \cos y \Rightarrow x = y$ (both acute)

2. Linear approximant: $L(x) = m(x-a) + y(a)$

2. Construct the linear approximant passing through the point $(e, 0)$ of $y(x)$ defined implicitly by

$$e^y - y^2 - \ln x = 0.$$

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$a = e \quad y(a) = y(e) = 0$

Does $(e,0)$ satisfy eq? : $e^0 - 0 - \ln e = 1 - 1 = 0$ ✓ Yes.

$m = $ slope $= y'(e)$. Implicit diff $\frac{d}{dx}\left(e^y - y^2 - \ln x\right) = 0 \Rightarrow$

$$\left.\begin{aligned} e^y y' - 2yy' - \frac{1}{x} &= 0 \\ \text{At } (x,y) &= (e,0) \end{aligned}\right\} \Rightarrow e^0 y'(e) - \frac{1}{e} = 0 \Rightarrow y'(e) = \frac{1}{e}$$

$$\Rightarrow L(x) = \frac{1}{e}(x-e).$$

3. 3. Determine the limit

$$L = \lim_{x\to\infty}\left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right).$$

$a = \sqrt{x^2+x+1} \quad b = \sqrt{x^2-x}$

$\infty - \infty$ Indeterminancy

$$L = \lim_{x\to\infty}(a-b) = \lim_{x\to\infty}\frac{a^2-b^2}{a+b} = \lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} \Rightarrow$$

$$a^2 - b^2 = x^2 + x + 1 - (x^2 - x) = 2x + 1$$

$$L = \lim_{x\to\infty}\frac{\cancel{x}\left(2+\frac{1}{x}\right)}{\cancel{x}\left[\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} + \sqrt{1-\frac{1}{x}}\right]} = 1$$

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4.

4. Evaluate the integral

$$I = \int \frac{x^2 - e^{2x}}{x + e^x}\,\mathrm{d}x.$$

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$$a = x^2 \quad b = e^{2x}$$

$$I = \int \frac{a^2 - b^2}{a+b}\,dx = \int (a-b)\,dx$$

$$I = \int (x - e^x)\,dx = \frac{1}{2}x^2 - e^x + C$$

5.

5. Evaluate the integral

$$I = \int \frac{2 + 3\cos x}{\sin^2 x}\,\mathrm{d}x.$$

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$$I = 2\int \frac{dx}{\sin^2 x} + 3\int \frac{\cos x}{\sin^2 x}\,dx$$

$$\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

$$\frac{d}{dx}\frac{1}{\sin x} = -\frac{\cos x}{\sin^2 x}$$

$\Rightarrow$

$$I = -2\cot x - \frac{3}{\sin x} + C.$$