Final Examination

Solve the following problems (5 course points each). Present a brief motivation of your method of solution. Problems 9 and 10 are optional; attempt them if you wish to improve your midterm examination score.

1. Form the matrix product corresponding to the following linear combinations

$$\boldsymbol{b}_1 = x_1\,\boldsymbol{a}_1 + x_2\,\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n,$$

$$\boldsymbol{b}_2 = y_1\,\boldsymbol{a}_1 + y_2\,\boldsymbol{a}_2 + \cdots + y_n\boldsymbol{a}_n,$$

$$\boldsymbol{b}_3 = z_1\,\boldsymbol{a}_1 + z_2\,\boldsymbol{a}_2 + \cdots + z_n\boldsymbol{a}_n\,.$$

Specify all matrix dimensions and column vectors.

Solution. Consider $\boldsymbol{a}_1 \in \mathbb{R}^m$. Consistency of vector addition then implies $\boldsymbol{a}_2, \ldots, \boldsymbol{a}_n, \boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3 \in \mathbb{R}^m$. Form

$$\boldsymbol{B} = [\ \boldsymbol{b}_1\ \ \boldsymbol{b}_2\ \ \boldsymbol{b}_3\ ] = \boldsymbol{A}\boldsymbol{C} \in \mathbb{R}^{m\times p}, p = 3.$$

The vectors entering into the linear combinations are

$$\boldsymbol{A} = [\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ \ \ldots\ \ \boldsymbol{a}_n\ ] \in \mathbb{R}^{m\times n}.$$

The scaling coefficients of the three linear combinations are

$$\boldsymbol{C} = [\ \boldsymbol{c}_1\ \ \boldsymbol{c}_2\ \ \boldsymbol{c}_3\ ] = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_n & y_n & z_n \end{bmatrix} \in \mathbb{R}^{n\times p}, p = 3.$$

2. For $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ let $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$ and $\boldsymbol{y} \neq \boldsymbol{0}$ be a solution of the linear system $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}$. Compute the angle between $\boldsymbol{b}$ and $\boldsymbol{y}$.

Solution. From $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$ deduce $\boldsymbol{b} \in C(\boldsymbol{A})$. From $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}$, deduce that $\boldsymbol{y} \in N(\boldsymbol{A}^T)$. The FTLA states $C(\boldsymbol{A}) \perp N(\boldsymbol{A}^T)$, hence $\boldsymbol{b} \perp \boldsymbol{y}$, and the angle between the two vectors is $\theta = \pi/2$ (orthogonal).

3. With $\boldsymbol{Q} \in \mathbb{R}^{m\times m}$ known to be orthogonal, carry out the following block matrix multiplication. Identify dimensions of all blocks, and the blocks and the dimensions of the resulting $\boldsymbol{C}$ matrix

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix}^T.$$

Solution. Consistency of multiplication requires $\boldsymbol{I}, \boldsymbol{0} \in \mathbb{R}^{m\times m}$, thereby leading to $\boldsymbol{C} \in \mathbb{R}^{2m\times 2m}$. Apply "row-over-columns" for matrix blocks, noting that $\boldsymbol{I}^T = \boldsymbol{I}$, $\boldsymbol{0}^T = \boldsymbol{0}$, $\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}$

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{Q}^T & \boldsymbol{0} \\ \boldsymbol{I} & \boldsymbol{Q}^T \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}\boldsymbol{Q}^T + \boldsymbol{I} & \boldsymbol{Q}^T \\ \boldsymbol{A} & \boldsymbol{A}\boldsymbol{Q}^T \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T + \boldsymbol{Q} + \boldsymbol{A} & \boldsymbol{I} + \boldsymbol{A}\boldsymbol{Q}^T \\ \boldsymbol{Q}\boldsymbol{A} & \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T \end{bmatrix}.$$

4. Compute $\boldsymbol{c} = \boldsymbol{A}^T\boldsymbol{b}$ and the projection of $\boldsymbol{b}$ onto $C(\boldsymbol{A})$ for

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ -1 & 2 \\ -4 & 0 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Solution. Apply "row-over-columns" rule to obtain

$$\boldsymbol{c}=\begin{bmatrix}2 & 3\\ -1 & 1\\ -1 & 2\\ -4 & 0\end{bmatrix}^T\begin{bmatrix}1\\ -1\\ -1\\ 1\end{bmatrix}=\begin{bmatrix}2 & -1 & -1 & -4\\ 3 & 1 & 2 & 0\end{bmatrix}\begin{bmatrix}1\\ -1\\ -1\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.$$

The above implies $\boldsymbol{b}\in N(\boldsymbol{A}^T)$, and by the FTLA the projection of $\boldsymbol{b}$ onto $C(\boldsymbol{A})$ is the zero vector.

5. Find the LU decomposition of

$$\boldsymbol{A}=\begin{bmatrix}1 & 1 & 1\\ 2 & 5 & 5\\ 4 & 9 & 15\end{bmatrix}.$$

Solution. Carry out reduction to upper triangular form, noting multipliers used in the process

$$\boldsymbol{L}_1\boldsymbol{A}=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -4 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 2 & 5 & 5\\ 4 & 9 & 15\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\ 0 & 3 & 3\\ 0 & 5 & 11\end{bmatrix}.$$

$$\boldsymbol{L}_2\boldsymbol{L}_1\boldsymbol{A}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -5/3 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & 3 & 3\\ 0 & 5 & 11\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\ 0 & 3 & 3\\ 0 & 0 & 6\end{bmatrix}=\boldsymbol{U}.$$

Find $\boldsymbol{A}=\boldsymbol{L}_1^{-1}\boldsymbol{L}_2^{-1}\boldsymbol{U}=\boldsymbol{L}\boldsymbol{U}$. Compute

$$\boldsymbol{L}=\boldsymbol{L}_1^{-1}\boldsymbol{L}_2^{-1}=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 4 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 5/3 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 4 & 5/3 & 1\end{bmatrix}.$$

Verify

$$\boldsymbol{L}\boldsymbol{U}=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 4 & 5/3 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & 3 & 3\\ 0 & 0 & 6\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\ 2 & 5 & 5\\ 4 & 9 & 15\end{bmatrix}.\checkmark$$

6. State the eigenvalues and eigenvectors of $\boldsymbol{C}=\boldsymbol{B}\boldsymbol{A}$, $\boldsymbol{A},\boldsymbol{B}\in\mathbb{R}^{2\times 2}$, with $\boldsymbol{C}$ the matrix describing: (a) rotation by $\theta=\pi/2$ ($\boldsymbol{A}$ matrix) followed by reflection across the x_1 axis ($\boldsymbol{B}$ matrix).

Solution. Let $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{q}$, $\boldsymbol{z}=\boldsymbol{B}\boldsymbol{y}=\boldsymbol{C}\boldsymbol{q}$. From sketch below, note that $\boldsymbol{q}_1=[\,1\ \ -1\,]^T$ rotated by $\pi/2$ becomes $\boldsymbol{y}_1=[\,1\ \ 1\,]^T$, which when reflected acorss the horizontal axis is again $\boldsymbol{q}_1=[\,1\ \ -1\,]^T$, thus an eigenvector with associated eigenvalue $\lambda_1=1$. Similarly, vector $\boldsymbol{q}_2=[\,1\ \ 1\,]^T$ rotated by $\pi/2$ becomes $\boldsymbol{y}_2=[\,-1\ \ 1\,]^T$, which when reflected acorss the horizontal axis is $[\,-1\ \ -1\,]^T=-\boldsymbol{q}_2$, thus an eigenvector with eigenvalue $\lambda_2=-1$.

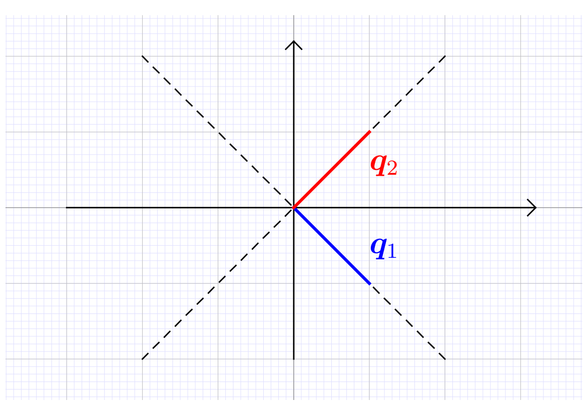

Figure 1.

7. Compute the eigendecomposition of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Solution. The characteristic polynomial is

$$p(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 2),$$

with resulting eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2$. For $\lambda_1 = 0$ perform row reduction to find eigenvector $\boldsymbol{x}_1$

$$\boldsymbol{A} - \lambda_1 \boldsymbol{I} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \boldsymbol{q}_1 = \boldsymbol{x}_1 / \|\boldsymbol{x}_1\| = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Similarly, for $\lambda_2 = 2$

$$\boldsymbol{A} - \lambda_2 \boldsymbol{I} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{q}_2 = \boldsymbol{x}_2 / \|\boldsymbol{x}_2\| = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since $\boldsymbol{A} = \boldsymbol{A}^T$, the eigendecomposition exists and is orthogonal

$$\boldsymbol{A} = \boldsymbol{Q} \boldsymbol{\Lambda} \boldsymbol{Q}^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

8. Find the SVD of

$$\boldsymbol{A} = \begin{bmatrix} 4 & 0 \\ 4 & 0 \end{bmatrix}.$$

Solution. Recall SVD $\boldsymbol{A} = \boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^T$, with $\boldsymbol{A}, \boldsymbol{\Sigma} \in \mathbb{R}^{m \times n}$, $\boldsymbol{U} \in \mathbb{R}^{m \times m}$ orthogonal, $\boldsymbol{V} \in \mathbb{R}^{n \times n}$ orthogonal. For this problem $m = n = 2$. Further recall $\boldsymbol{U} = [\, \boldsymbol{u}_1 \ \ldots \ \boldsymbol{u}_r \ \boldsymbol{u}_{r+1} \ \ldots \ \boldsymbol{u}_m \,], \boldsymbol{V} = [\, \boldsymbol{v}_1 \ \ldots \ \boldsymbol{v}_r \ \boldsymbol{v}_{r+1} \ \ldots \ \boldsymbol{v}_n \,]$. The matrix $\boldsymbol{A}$ has rank $r = 1$, and $\boldsymbol{u}_1$ can be taken as

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since $\boldsymbol{U}$ is orthogonal take

$$\boldsymbol{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

to obtain

$$\boldsymbol{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Take $\boldsymbol{V} = \boldsymbol{I}$ to obtain

$$\boldsymbol{A}\boldsymbol{V} = \boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma} \Rightarrow \begin{bmatrix} 4 & 0 \\ 4 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_1/\sqrt{2} & 0 \\ \sigma_1/\sqrt{2} & 0 \end{bmatrix}.$$

Deduce that $\sigma_1 = 4\sqrt{2}$, completing the SVD.

9. Form the matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{2 \times 2}$, $\boldsymbol{C} = \boldsymbol{B}\boldsymbol{A}$, where $\boldsymbol{C}$ is the matrix describing: (a) rotation by $\theta = \pi/2$ ($\boldsymbol{A}$ matrix) followed by reflection across the x_1 axis ($\boldsymbol{B}$ matrix).

Solution. The matrices are

$$\boldsymbol{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

10. Find bases for the four fundamental spaces of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & -1 & -4 \\ 3 & 1 & 2 & 0 \end{bmatrix}.$$

Solution. Note that $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with $m=3$, $n=4$. Carry out row reduction

$$\boldsymbol{A} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & -7 & -12 \\ 0 & -5 & -7 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & -7 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

to find $r=\operatorname{rank}(\boldsymbol{A})=2$.

$C(\boldsymbol{A})$: FTLA states $\dim C(\boldsymbol{A})=r=2$. Take $r=2$ linearly independent columns as the basis, e.g.,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

$C(\boldsymbol{A}^T)$: FTLA states $\dim C(\boldsymbol{A}^T)=r=2$. Take $r=2$ linearly independent rows as the basis, e.g.,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \\ -4 \end{bmatrix} \right\}.$$

$N(\boldsymbol{A}^T)$: FTLA states $\dim N(\boldsymbol{A}^T)=m-r=1$. From row reduction of $\boldsymbol{A}^T$

$$\boldsymbol{A}^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -7 & -7 \\ 0 & -12 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

consider system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}.$$

Take $x_3=\lambda$ as a free parameter to obtain

$$\begin{cases} x_1 + 2x_2 = -3\lambda \\ x_2 = -\lambda \end{cases}, x_1 = x_2 = -\lambda.$$

A basis vector for $N(\boldsymbol{A}^T)$ is therefore

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Verify

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \checkmark.$$

$N(\boldsymbol{A})$: FTLA states $\dim N(\boldsymbol{A})=n-r=2$. Continue above row reduction of $\boldsymbol{A}$ to obtain reduced row echelon form

$$\boldsymbol{A} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & -7 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 7/5 & 12/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/5 & -4/5 \\ 0 & 1 & 7/5 & 12/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and form system

$$\begin{cases} 5x_1 + x_3 - 4x_4 = 0 \\ 5x_2 + 7x_3 + 12x_4 = 0 \end{cases}.$$

Consider $x_3 = \lambda$, $x_4 = \mu$ to be free parameters to obtain

$$\begin{cases} x_1 = -(\lambda - 4\mu)/5 \\ x_2 = -(7\lambda + 12\mu)/5 \end{cases}.$$

For $x_4 = \mu = 0$ obtain

$$x_1 = -\frac{1}{5}\lambda, x_2 = -\frac{7}{5}\lambda, x_3 = \lambda, x_4 = 0,$$

For $x_3 = \lambda = 0$ obtain

$$x_1 = \frac{4}{5}\mu, x_2 = -\frac{12}{5}\mu, x_3 = 0, x_4 = \mu.$$

Deduce that a basis set for $N(\boldsymbol{A})$ is

$$\left\{ \begin{bmatrix} -1 \\ -7 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -12 \\ 0 \\ 5 \end{bmatrix} \right\}.$$

Verify

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & -1 & -4 \\ 3 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ -7 & -12 \\ 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$