MATH347.SP.01 Midterm Practice Examination

Instructions. *Answer the following questions. Provide a motivation of your approach and the reasoning underlying successive steps in your solution. Write neatly and avoid erasures. Use scratch paper to sketch out your answer for yourself, and then transcribe your solution to the examination you turn in for grading. Illegible answers are not awarded any credit. Presentation of calculations without mention of the motivation and reasoning are not awarded any credit. Each complete, correct solution to an examination question is awarded 3 course grade points. Your primary goal should be to demonstrate understanding of course topics and skill in precise mathematical formulation and solution procedures.*

1. Consider a vector $\boldsymbol{b}\in\mathbb{R}^{3\times 1}$ with components $b_1=1$, $b_2=2$, $b_3=-1$ in the canonical basis

$$\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

Let x_1,x_2,x_3 denote the components of $\boldsymbol{b}$ with respect to the basis

$$\{\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3\}=\left\{\begin{pmatrix}1\\-1\\2\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right\}.$$

Compute $\boldsymbol{x}=[\ x_1\ \ x_2\ \ x_3\ ]^T\in\mathbb{R}^{3\times 1}$. (3 points)

Solution. Problem asks for solution of system $\boldsymbol{b}=\boldsymbol{A}\boldsymbol{x}$

$$\boldsymbol{b}=\begin{bmatrix}1\\2\\-1\end{bmatrix}=\boldsymbol{A}\boldsymbol{x}=\begin{bmatrix}1&0&2\\-1&1&1\\2&2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}.$$

Form bordered matrix and bring to upper trinagular form (Gauss elimination)

$$\begin{bmatrix}1&0&2&1\\-1&1&1&2\\2&2&1&-1\end{bmatrix}\sim\begin{bmatrix}1&0&2&1\\0&1&3&3\\0&2&-3&-3\end{bmatrix}\sim\begin{bmatrix}1&0&2&1\\0&1&3&3\\0&0&-9&-9\end{bmatrix}$$

By back substitution, find $x_3=1$, $x_2=0$, $x_1=-1$

2. Let

$$\boldsymbol{A}=\begin{bmatrix}1&2\\2&1\\3&1\end{bmatrix}$$

a) Find bases for the subspaces $C(\boldsymbol{A})$, $C(\boldsymbol{A}^T)$, $N(\boldsymbol{A})$, $N(\boldsymbol{A}^T)$. (4 points)

b) Compute the orthogonal projection of the vector $\boldsymbol{b} = [\ 1\ \ 1\ \ 1\ ]^T$ onto $C(\boldsymbol{A})$. (2 points)

c) Compute the QR decomposition of $\boldsymbol{A}$. (3 points)

Solution. a) Denote $\boldsymbol{A} = [\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ ]$. Since $\boldsymbol{a}_1, \boldsymbol{a}_2$ are linearly independent, a basis for $C(\boldsymbol{A})$ is $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ and $\operatorname{rank}(\boldsymbol{A}) = 2 = \dim C(\boldsymbol{A}) = \dim C(\boldsymbol{A}^T)$. A basis for the row space is

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

The null space is $N(\boldsymbol{A}) = \{\boldsymbol{0}\}$, with $\dim N(\boldsymbol{A}) = 0$, empty basis set. For $N(\boldsymbol{A}^T)$ consider the system

$$\boldsymbol{A}^T \boldsymbol{y} = \boldsymbol{0} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -5 & 0 \end{bmatrix}$$

with solutions

$$\boldsymbol{y} = \begin{bmatrix} \left(-3 + \frac{10}{3}\right) y_3 \\ -\frac{5}{3} y_3 \\ y_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} y_3.$$

A basis for $N(\boldsymbol{A}^T)$ is

$$\{\boldsymbol{w}\} = \left\{ \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \right\}$$

b) To conpute the projection $\boldsymbol{y}$, subtract from $\boldsymbol{b}$ its component along $\boldsymbol{w}$ determined above (FTLA states $\boldsymbol{b} = \boldsymbol{y} + \boldsymbol{z}$, with $\boldsymbol{y} \in C(\boldsymbol{A})$, $\boldsymbol{z} \in N(\boldsymbol{A}^T)$)

$$\text{Let } \boldsymbol{v} = \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \frac{1}{\sqrt{35}} \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}, \boldsymbol{z} = (\boldsymbol{v}^T \boldsymbol{b}) \boldsymbol{v} = -\frac{1}{35} \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \Rightarrow \boldsymbol{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{35} \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 36 \\ 30 \\ 38 \end{bmatrix}.$$

c) Find the $\boldsymbol{QR}$-decomposition

$$\boldsymbol{q}_1 = \frac{\boldsymbol{a}_1}{\|\boldsymbol{a}_1\|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$$\boldsymbol{q}_2 = \boldsymbol{a}_2 - (\boldsymbol{q}_1^T \boldsymbol{a}_2) \boldsymbol{q}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{14}(7) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \\ -1/2 \end{bmatrix}, \|\boldsymbol{q}_2\| = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} & 3/\sqrt{10} \\ 2/\sqrt{14} & 0 \\ 3/\sqrt{14} & -1/\sqrt{10} \end{bmatrix} \begin{bmatrix} \sqrt{14} & 7/\sqrt{14} \\ 0 & \sqrt{10} \end{bmatrix}$$