MATH347: Linear algebra for applications

Homework 10

This assignment is a worksheet of exercises intended as preparation for the Final Examination. You should:

1. Review Lessons 1 to 12
2. Set aside 60 minutes to solve these exercises. Each exercise is meant to be solved within 3 minutes. If you cannot find a solution within 3 minutes, skip to the next one.
3. Check your answers in Matlab. Revisit theory for skipped or incorrectly answered exercies.
4. Turn in a PDF with your brief handwritten answers that specify your motivation, approach, calculations, answer. It is good practice to start all answers by briefly recounting the applicable definitions.

1 Vector operations

1. Find the linear combination of vectors $\boldsymbol{u} = [\ 1\ \ 1\ \ 1\ ]$, $\boldsymbol{v} = [\ 1\ \ 2\ \ 3\ ]$ with scaling coefficients $\alpha = 2$, $\beta = 1$.
2. Express the above linear combination $\boldsymbol{b}$ as a matrix-vector product $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$. Define $\boldsymbol{x}$ and the column vectors of $\boldsymbol{A} = [\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ ]$.
3. Consider $\boldsymbol{u} = [\ 1\ \ 1\ \ 0\ ]^T$, $\boldsymbol{v} = [\ 1\ \ 1\ \ 1\ ]^T$. Compute the 2-norms of $\boldsymbol{u}, \boldsymbol{v}$. Determine the angle between $\boldsymbol{u}, \boldsymbol{v}$.
4. Consider $\boldsymbol{u} = [\ 1\ \ 1\ \ 0\ ]^T$, $\boldsymbol{v} = [\ 1\ \ 1\ \ 1\ ]^T$. Define vector $\boldsymbol{w}$ such that $\boldsymbol{v} + \boldsymbol{w}$ is orthogonal to $\boldsymbol{u}$. Write the equation to determine $\boldsymbol{w}$, and then compute $\boldsymbol{w}$.
5. Determine $\boldsymbol{q}_1, \boldsymbol{q}_2$ to be of unit norm and in the direction of vectors $\boldsymbol{u}, \boldsymbol{w}$ from Ex. 4. Form $\hat{\boldsymbol{Q}} = [\ \boldsymbol{q}_1\ \ \boldsymbol{q}_2\ ]$. Compute $\hat{\boldsymbol{Q}}\hat{\boldsymbol{Q}}^T$ and $\hat{\boldsymbol{Q}}^T\hat{\boldsymbol{Q}}$.
6. Determine vector $\boldsymbol{q}_3$ orthonormal to vectors $\boldsymbol{q}_1, \boldsymbol{q}_2$ from Ex. 5.
7. Establish whether vectors $\boldsymbol{u} = [\ 1\ \ 2\ \ 3\ ]^T$, $\boldsymbol{v} = [\ -3\ \ 1\ \ -2\ ]^T$, $\boldsymbol{w} = [\ 2\ \ -3\ \ 1\ ]^T$ all lie in the same plane within $\mathbb{R}^3$.
8. Determine $\boldsymbol{v}$ the reflection of vector $\boldsymbol{u} = [\ 1\ \ \sqrt{3}\ ]^T$ across vector $\boldsymbol{w} = [\ 1\ \ 1\ ]^T$.
9. Determine $\boldsymbol{w}$ the rotation of vector $\boldsymbol{u} = [\ 1\ \ \sqrt{3}\ ]^T$ by angle $\theta = -\pi/6$.
10. Compute $\boldsymbol{z} = \boldsymbol{v} - \boldsymbol{w}$ with $\boldsymbol{v}, \boldsymbol{w}$ from Ex. 8,9.

2 Matrix operations

1. Find two linear combinations of vectors $\boldsymbol{u} = [\ 1\ \ 1\ \ 1\ ]$, $\boldsymbol{v} = [\ 1\ \ 2\ \ 3\ ]$ first with scaling coefficients $\alpha = 2$, $\beta = 1$, and then with scaling coefficients $\alpha = 1$, $\beta = 2$.
2. Express the above linear combinations $\boldsymbol{B}$ as a matrix-matrix product $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{X}$. Define the column vectors of $\boldsymbol{A}, \boldsymbol{X}$.
3. Consider $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times m}$. Which of the following matrices are always equal to $\boldsymbol{C} = (\boldsymbol{A} - \boldsymbol{B})^2$?
   a) $\boldsymbol{A}^2 - \boldsymbol{B}^2$
   b) $(\boldsymbol{B} - \boldsymbol{A})^2$
   c) $\boldsymbol{A}^2 - 2\boldsymbol{A}\boldsymbol{B} + \boldsymbol{B}^2$
   d) $\boldsymbol{A}(\boldsymbol{A} - \boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{B} - \boldsymbol{A})$
   e) $\boldsymbol{A}^2 - \boldsymbol{A}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{A} + \boldsymbol{B}^2$
4. Find the inverse of
$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$
5. Verify that the inverse of $\boldsymbol{A} = \boldsymbol{I} - \boldsymbol{u}\boldsymbol{v}^T$ is
$$\boldsymbol{A}^{-1} = \boldsymbol{I} + \frac{\boldsymbol{u}\boldsymbol{v}^T}{1 - \boldsymbol{v}^T\boldsymbol{u}}$$

when $\boldsymbol{v}^T \boldsymbol{u} \neq 1$.

6. Find $\boldsymbol{A}^T, \boldsymbol{A}^{-1}, (\boldsymbol{A}^{-1})^T, (\boldsymbol{A}^T)^{-1}$ for

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix}.$$

7. Describe within $\mathbb{R}^3$ the geometry of the column spaces of matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

8. The vector subspaces of $\mathbb{R}^2$ are lines, $\mathbb{R}^2$ itself and $Z = \{[\,0 \;\; 0\,]^T\}$. What are the vector subspaces of $\mathbb{R}^3$?

9. Reduce the following matrices to row echelon form

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

10. Determine the null space of

$$\boldsymbol{A} = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix}.$$