Lesson concepts:

- Vectors
- Vector operations
- Linear combinations
- Matrix vector multiplication

- Some quantities arising in applications can be expressed as single numbers, called "scalars"
  - Speed of a car on a highway $v = 35$ mph
  - A person's height $H = 183$ cm
- Many other quantitites require more than one number:
  - Position in a city: "Intersection of 86th St and 3rd Av"
  - Position in 3D space: (x, y, z)
  - Velocity in 3D space: (u, v, w)

Definition. *A* *vector* *is a grouping of* m *scalars*

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \in S^m, v_i \in S$$

- The scalars usually are naturals ($S = \mathbb{N}$), integers ($S = \mathbb{Z}$), rationals ($S = \mathbb{Q}$), reals ($S = \mathbb{R}$), or complex numbers ($S = \mathbb{C}$)
- We often denote the dimension and set of scalars as $\boldsymbol{v} \in S^m$, e.g. $\boldsymbol{v} \in \mathbb{R}^m$
- Sets of vectors are denoted as

$$\mathcal{V} = \left\{ \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}, v_i \in S \right\} \tag{1}$$

- A vector can also be interpreted as a function from a subset of $\mathbb{N}$ to S

$$v: \{1, 2, .., m\} \rightarrow S$$

- **Vector addition**. Consider two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$. We define the sum of the two vectors as the vector containing the sum of the components

$$\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_m + v_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

```
>> u=[1 2 3]; v=[-2 1 2]; u+v
```

- **Scalar multiplication**. Consider $\alpha \in S$, $\boldsymbol{u} \in \mathcal{V}$. We define the multiplication of vector $\boldsymbol{u}$ by scalar α as the vector containing the product of each component of $\boldsymbol{u}$ with the scalar α

$$\boldsymbol{w} = \alpha\ \boldsymbol{u} = \begin{bmatrix} \alpha\ u_1 \\ \alpha\ u_2 \\ \vdots \\ \alpha\, u_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

- **Vector addition**. Consider two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$. We define the sum of the two vectors as the vector containing the sum of the components

$$\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_m + v_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

```
>> u=[1 2 3]; v=[-2 1 2]; u+v
```

$$\left(\begin{array}{ccc} -1 & 3 & 5 \end{array} \right)$$

- **Scalar multiplication**. Consider $\alpha \in S$, $\boldsymbol{u} \in \mathcal{V}$. We define the multiplication of vector $\boldsymbol{u}$ by scalar α as the vector containing the product of each component of $\boldsymbol{u}$ with the scalar α

$$\boldsymbol{w} = \alpha\ \boldsymbol{u} = \begin{bmatrix} \alpha\ u_1 \\ \alpha\ u_2 \\ \vdots \\ \alpha\, u_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

- **Linear combination**. Let $\alpha,\beta\in S$, $\boldsymbol{u},\boldsymbol{v}\in\mathcal{V}$. Define a linear combination of two vectors by

$$\boldsymbol{w}=\alpha\ \boldsymbol{u}+\beta\ \boldsymbol{v}=\begin{bmatrix}\alpha\,u_1\\ \alpha\ u_2\\ \vdots\\ \alpha\ u_m\end{bmatrix}+\begin{bmatrix}\beta\ v_1\\ \beta\ v_2\\ \vdots\\ b\beta\,v_m\end{bmatrix}=\begin{bmatrix}\alpha\ u_1+\beta\ v_1\\ \alpha\ u_2+\beta\ v_2\\ \vdots\\ \alpha\,u_m+\beta\,v_m\end{bmatrix}=\begin{bmatrix}w_1\\ w_2\\ \vdots\\ w_m\end{bmatrix}$$

- Linear combination of n vectors

$$\boldsymbol{b}=x_1\boldsymbol{a}_1+x_2\boldsymbol{a}_2+\cdots+x_n\boldsymbol{a}_n=\begin{bmatrix}x_1a_{11}+x_2a_{12}+\cdots+x_na_{1n}\\ x_1a_{21}+x_2a_{22}+\cdots+x_na_{2n}\\ \vdots\\ x_1a_{m1}+x_2a_{m2}+\cdots+x_na_{mn}\end{bmatrix}$$

"Start at the center of town. Go east 3 blocks and north 2 blocks. What is your final position?"

$$\boldsymbol{s} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{e}_E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{e}_N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\alpha_E = 3, \alpha_N = 2$$

$$\boldsymbol{f} = \boldsymbol{s} + \alpha_E \boldsymbol{e}_E + \alpha_N \boldsymbol{e}_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Linear combinations allow us to express a position in space using a standard set of directions. Questions:

- How many standard directions are needed?
- Can any position be specified as a linear combination?
- How to find the scalars needed to express a position as a linear combination?

Seek a more compact notation for the linear combination

$$a\begin{bmatrix}1\\2\\3\end{bmatrix}+b\begin{bmatrix}-1\\0\\1\end{bmatrix}+c\begin{bmatrix}2\\0\\1\end{bmatrix}=\begin{bmatrix}a-b+2c\\2a\\3a+b+c\end{bmatrix}$$

- Group the vectors together to form a "matrix"

$$\boldsymbol{A}=\begin{bmatrix}1&-1&2\\2&0&0\\3&1&1\end{bmatrix}$$

- Group the scalars together to form a vector

$$\boldsymbol{u}=\begin{bmatrix}a\\b\\c\end{bmatrix}$$

- Define matrix-vector multiplication

$$\boldsymbol{A}\boldsymbol{u} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a - b + 2c \\ 2a \\ 3a + b + c \end{bmatrix}$$

- In general

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \begin{bmatrix} x_1 a_{11} + x_2 a_{12} + \cdots + x_n a_{1n} \\ x_1 a_{21} + x_2 a_{22} + \cdots + x_n a_{2n} \\ \vdots \\ x_1 a_{m1} + x_2 a_{m2} + \cdots + x_n a_{mn} \end{bmatrix}$$

- Construct linear combination of vectors $\boldsymbol{u} = [\,1 \;\; -1 \;\; 2\,]$, $\boldsymbol{v} = [\,2 \;\; 1 \;\; -1\,]$ scaled by $\alpha = 2$ and $\beta = 3$, respectively

```
>> u=[1 -1 2]; v=[2 1 -1]; alpha=2; beta=3;
>> alpha*u+beta*v
```

- Construct linear combination of vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ scaled by $\alpha = 2$ and $\beta = 3$, respectively

```
>> u=[1; -1; 2]; v=[2; 1; -1]; alpha*u+beta*v
>>
```

- Construct linear combination of vectors $\boldsymbol{u} = [\,1 \;\; -1 \;\; 2\,]$, $\boldsymbol{v} = [\,2 \;\; 1 \;\; -1\,]$ scaled by $\alpha = 2$ and $\beta = 3$, respectively

```
>> u=[1 -1 2]; v=[2 1 -1]; alpha=2; beta=3;
>> alpha*u+beta*v
```

(8 1 1)

- Construct linear combination of vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ scaled by $\alpha = 2$ and $\beta = 3$, respectively

```
>> u=[1; -1; 2]; v=[2; 1; -1]; alpha*u+beta*v
>>
```

- Construct linear combination of vectors $\boldsymbol{u} = [\, 1 \;\; -1 \;\; 2 \,]$, $\boldsymbol{v} = [\, 2 \;\; 1 \;\; -1 \,]$ scaled by $\alpha = 2$ and $\beta = 3$, respectively

```
>> u=[1 -1 2]; v=[2 1 -1]; alpha=2; beta=3;
>> alpha*u+beta*v
```

$$\begin{pmatrix} 8 & 1 & 1 \end{pmatrix}$$

- Construct linear combination of vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ scaled by $\alpha = 2$ and $\beta = 3$, respectively

```
>> u=[1; -1; 2]; v=[2; 1; -1]; alpha*u+beta*v
```

$$\begin{pmatrix} 8 \\ 1 \\ 1 \end{pmatrix}$$

```
>>
```

- Construct linear combination of vectors $\boldsymbol{u}=\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\boldsymbol{v}=\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ scaled by $\alpha=2$ and $\beta=3$, respectively

```
>> u=[1; -1; 2]; v=[2; 1; -1]; alpha=2; beta=3;
>> A=[u v]; x=[alpha; beta]; A*x
>>
```

- Construct linear combination of vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ scaled by $\alpha = 2$ and $\beta = 3$, respectively

```
>> u=[1; -1; 2]; v=[2; 1; -1]; alpha=2; beta=3;
```

```
>> A=[u v]; x=[alpha; beta]; A*x
```

$$\begin{pmatrix} 8 \\ 1 \\ 1 \end{pmatrix}$$

```
>>
```