- Previous concept synopsis:
  - Vectors are groupings of scalars
  - Vector addition and multiplication of a vector by a scalar have been defined
  - Linear combination defined
  - Matrices are groupings of vectors
  - Matrix-vector product expresses a linear combination in a concise notation
- New concepts:
  - Scalar product of two vectors
  - Orthogonal vectors
  - Norm of a vector

Definition. *An* m *by* n *matrix is a grouping of* n *vectors,*

$$\boldsymbol{A} = [\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ \ \dots\ \ \boldsymbol{a}_n\ ] \in S^{m\times n},$$

where each vector has m *scalar components* $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n \in S^m$.

- Notation conventions:
  - scalars: normal face, Latin or Greek letters, $a, b, \alpha, \beta, u_1, a_{11}, A_{11}, I_{12}$
  - vectors: bold face, lower case Latin letters, $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}_1$
  - matrices: bold face, upper case Latin letters, $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{L}_1$

- Matrix components

$$\boldsymbol{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \boldsymbol{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \boldsymbol{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \Rightarrow$$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A real-valued matrix with m lines and n: $\boldsymbol{A} \in \mathbb{R}^{m \times n}$

```
>> A=[3 1 2; -1 0 1; 3 4 1]; disp(A)
```

- Matrix components

$$\boldsymbol{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \boldsymbol{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \boldsymbol{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \Rightarrow$$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A real-valued matrix with m lines and n: $\boldsymbol{A} \in \mathbb{R}^{m \times n}$

```
>> A=[3 1 2; -1 0 1; 3 4 1]; disp(A)
```

```
   3   1   2
  -1   0   1
   3   4   1
```

- Instead of explicitly writing out components, it is often convenient to specify a matrix by a rule to construct each component

$$\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{A} = [a_{ij}]$$

with indices taking values $i \in \{1, ..., m\}, j \in \{1, ..., n\}$.

Example: A *Hilbert matrix* $\boldsymbol{H}_m$ is defined as

$$\boldsymbol{H}_m \in \mathbb{Q}^{m \times m}, \boldsymbol{H}_m = \left[\frac{1}{i+j-1} \right]$$

- Note that a vector is a matrix with a single column. The notation $\boldsymbol{v} \in \mathbb{R}^m$, is a customary shorter form of $\boldsymbol{v} \in \mathbb{R}^{m \times 1}$.

- Scalar products are useful in many other contexts than real-valued vectors.

Definition. *Consider vectors* $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$ *and scalar* $a \in \mathbb{R}$. *The function*

$$\langle\,,\rangle : \mathcal{V} \times \mathcal{V} \to S$$

is a scalar product if:

1. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$ *(Symmetry)*
2. $\langle a\boldsymbol{u}, \boldsymbol{v} \rangle = a\langle \boldsymbol{u}, \boldsymbol{v} \rangle, \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ *(Linearity in first argument)*
3. $\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geqslant 0, \langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 \Rightarrow \boldsymbol{u} = \boldsymbol{0}$ *(Positive definiteness)*

- Inner product of $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^m$, $\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \cdots + u_m v_m$

Definition. *The norm of a vector is a function that takes a vector argument, returns a positive real number,* $\|\ \|: \mathcal{V} \to \mathbb{R}_+$*, and for* $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$*,* $a \in S$*,satisfies properties:*

1. $\|a\boldsymbol{u}\| = |a|\,\|\boldsymbol{u}\|$
2. $\|\boldsymbol{u} + \boldsymbol{v}\| \leqslant \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$
3. $\|\boldsymbol{u}\| = 0 \Rightarrow \boldsymbol{u} = \boldsymbol{0}$.

- A norm embodies the concept of measurement of the magnitude of a vector
- Different ways of measuring the magnitude of a vector are most appropriate in various applications, resulting in different definitions of a vector norm for $\boldsymbol{u} \in \mathbb{R}^m$

1-norm	$\|\boldsymbol{u}\|_1 = \sum_{i=1}^m \|u_i\|$
2-norm (Euclidean norm)	$\|\boldsymbol{u}\|_2 = (\sum_{i=1}^m u_i^2)^{1/2} = (\boldsymbol{u}^T\boldsymbol{u})^{1/2}$
inf-norm	$\|\boldsymbol{u}\|_\infty = \max_{i\in\{1,2,\ldots,m\}} \|u_i\|$

- Different norms are distinguished by subscripts as above. The most commonly used norm is the Euclidean norm that corresponds to the square root of the scalar product, i.e. $\|\boldsymbol{u}\| = (\boldsymbol{u} \cdot \boldsymbol{u})^{1/2}$, in which case the subscript is often suppressed to simplify notation $\|\boldsymbol{u}\| = \|\boldsymbol{u}\|_2$

- Law of cosines

$$\|\boldsymbol{v}-\boldsymbol{w}\|^2=\|\boldsymbol{v}\|^2+\|\boldsymbol{w}\|^2-2\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|\cos(\theta)$$

- Dot product properties

$$\|\boldsymbol{v}-\boldsymbol{w}\|^2=\|\boldsymbol{v}\|^2+\|\boldsymbol{w}\|^2-2(\boldsymbol{v}\cdot\boldsymbol{w})$$

- Combine, and define

$$\cos(\theta)=\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|}$$

- In particular if $\cos(\theta)=0\Rightarrow\boldsymbol{v}\cdot\boldsymbol{w}=0$, $\boldsymbol{v},\boldsymbol{w}$ are said to be orthogonal

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = [\, \boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \boldsymbol{a}_3 \,]$$

```
>> A=[3 1 2; -1 0 1; 3 4 1]; disp(A)
```

```
>> A(:,2)
```

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = [\, \boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \boldsymbol{a}_3 \,]$$

```
>> A=[3 1 2; -1 0 1; 3 4 1]; disp(A)
```

```
     3     1     2
    -1     0     1
     3     4     1
```

```
>> A(:,2)
```

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

```
>> A=[3 1 2; -1 0 1; 3 4 1]; disp(A)
```

```
  3   1   2
 -1   0   1
  3   4   1
```

```
>> A(:,2)
```

$$\begin{pmatrix} 1 \\ 0.0 \\ 4 \end{pmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix}$$

```
>> A=[3 1 2; -1 0 1; 3 4 1]; disp(A)
```

```
>> A(2,:)
```

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

```
>> A=[3 1 2; -1 0 1; 3 4 1]; disp(A)
```

```
  3   1   2
 -1   0   1
  3   4   1
```

```
>> A(2,:)
```

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \left[\; \boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3 \; \right]$$

```
>> A=[3 1 2; -1 0 1; 3 4 1]; disp(A)
```

```
   3   1   2
  -1   0   1
   3   4   1
```

```
>> A(2,:)
```

$$\begin{pmatrix} 1 \\ 0.0 \\ 4 \end{pmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix}$$

```
>> A=[3 1 2; -1 0 1; 3 4 1]; disp(A)
```

```
>> A(:,2:3)
```

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix}$$

```
>> A=[3 1 2; -1 0 1; 3 4 1]; disp(A)
```

```
   3   1   2
  -1   0   1
   3   4   1
```

```
>> A(:,2:3)
```

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = [\, a_1 \;\; a_2 \;\; a_3 \,]$$

```
>> A=[3 1 2; -1 0 1; 3 4 1]; disp(A)
```

```
   3   1   2
  -1   0   1
   3   4   1
```

```
>> A(:,2:3)
```

$$\begin{pmatrix} 1 & 2 \\ 0.0 & 1 \\ 4 & 1 \end{pmatrix}$$

Example: Inner product of two vectors

```
>> u=[1 2 3]; v=[3 0 1]; dot(u,v)
```

```
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; dot(e1,e2)
```

```
>> dot(e1,e3)
```

```
>> dot(e2,e3)
```

```
>> dot(e1,e1)
```

```
>> dot(e2,e2)
```

```
>> dot(e3,e3)
```

Example: Inner product of two vectors

```
>> u=[1 2 3]; v=[3 0 1]; dot(u,v)
```

6

```
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; dot(e1,e2)
```

```
>> dot(e1,e3)
```

```
>> dot(e2,e3)
```

```
>> dot(e1,e1)
```

```
>> dot(e2,e2)
```

```
>> dot(e3,e3)
```

```
>> u=[1 2 3]; v=[3 0 1]; dot(u,v)
```

6

```
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; dot(e1,e2)
```

0.0

```
>> dot(e1,e3)
```

```
>> dot(e2,e3)
```

```
>> dot(e1,e1)
```

```
>> dot(e2,e2)
```

```
>> dot(e3,e3)
```

```
>> u=[1 2 3]; v=[3 0 1]; dot(u,v)
```

6

```
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; dot(e1,e2)
```

0.0

```
>> dot(e1,e3)
```

0.0

```
>> dot(e2,e3)
```

```
>> dot(e1,e1)
```

```
>> dot(e2,e2)
```

```
>> dot(e3,e3)
```

```
>> u=[1 2 3]; v=[3 0 1]; dot(u,v)
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6

```
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; dot(e1,e2)
```

0.0

```
>> dot(e1,e3)
```

0.0

```
>> dot(e2,e3)
```

0.0

```
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```

```
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```

0.0

```
>> dot(e1,e3)
```

0.0

```
>> dot(e2,e3)
```

0.0

```
>> dot(e1,e1)
```

1

```
>> dot(e2,e2)
```

```
>> dot(e3,e3)
```

```
>> u=[1 2 3]; v=[3 0 1]; dot(u,v)
```

6

```
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; dot(e1,e2)
```

0.0

```
>> dot(e1,e3)
```

0.0

```
>> dot(e2,e3)
```

0.0

```
>> dot(e1,e1)
```

1

```
>> dot(e2,e2)
```

1

```
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```

```
>> u=[1 2 3]; v=[3 0 1]; dot(u,v)
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```
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; dot(e1,e2)
```

0.0

```
>> dot(e1,e3)
```

0.0

```
>> dot(e2,e3)
```

0.0

```
>> dot(e1,e1)
```

1

```
>> dot(e2,e2)
```

1

```
>> dot(e3,e3)
```

1

```
>> u=[1 2 3]; v=[3 0 1]; norm(u)
```

```
>> sqrt(dot(u,u))
```

```
>> norm(v)
```

```
>> sqrt(dot(v,v))
```

```
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; norm(e1)
```

```
>> norm(e2)
```

```
>> norm(e3)
```

Example: Vector norm

```
>> u=[1 2 3]; v=[3 0 1]; norm(u)
```

3.7417

```
>> sqrt(dot(u,u))
```

```
>> norm(v)
```

```
>> sqrt(dot(v,v))
```

```
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; norm(e1)
```

```
>> norm(e2)
```

```
>> norm(e3)
```

Example: Vector norm

```
>> u=[1 2 3]; v=[3 0 1]; norm(u)
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```
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```
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```
>> sqrt(dot(v,v))
```

```
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; norm(e1)
```

```
>> norm(e2)
```

```
>> norm(e3)
```

```
>> u=[1 2 3]; v=[3 0 1]; norm(u)
```

3.7417

```
>> sqrt(dot(u,u))
```

3.7417

```
>> norm(v)
```

3.1623

```
>> sqrt(dot(v,v))
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; norm(e1)
>> norm(e2)
>> norm(e3)
```

Example: Vector norm

```
>> u=[1 2 3]; v=[3 0 1]; norm(u)
```

3.7417

```
>> sqrt(dot(u,u))
```

3.7417

```
>> norm(v)
```

3.1623

```
>> sqrt(dot(v,v))
```

3.1623

```
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; norm(e1)
```

```
>> norm(e2)
```

```
>> norm(e3)
```

```
>> u=[1 2 3]; v=[3 0 1]; norm(u)
```

3.7417

```
>> sqrt(dot(u,u))
```

3.7417

```
>> norm(v)
```

3.1623

```
>> sqrt(dot(v,v))
```

3.1623

```
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; norm(e1)
```

1

```
>> norm(e2)
```

```
>> norm(e3)
```

```
>> u=[1 2 3]; v=[3 0 1]; norm(u)
```

3.7417

```
>> sqrt(dot(u,u))
```

3.7417

```
>> norm(v)
```

3.1623

```
>> sqrt(dot(v,v))
```

3.1623

```
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; norm(e1)
```

1

```
>> norm(e2)
```

1

```
>> norm(e3)
```

```
>> u=[1 2 3]; v=[3 0 1]; norm(u)
```

3.7417

```
>> sqrt(dot(u,u))
```

3.7417

```
>> norm(v)
```

3.1623

```
>> sqrt(dot(v,v))
```

3.1623

```
>> e1=[1 0 0]; e2=[0 1 0]; e3=[0 0 1]; norm(e1)
```

1

```
>> norm(e2)
```

1

```
>> norm(e3)
```

1

Example: Angle between two vectors

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|}$$

```
>> dot(e1,e2)/norm(e1)/norm(e2)
```

```
>> acos(0.0)
```

```
>> dot(e1,e1)/norm(e1)/norm(e1)
```

```
>> acos(1)
```

```
>> u=[1 0]; v=[1 1]; dot(u,v)/norm(u)/norm(v)
```

```
>> acos(0.7071)/pi*180
```

Example: Angle between two vectors

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|}$$

```
>> dot(e1,e2)/norm(e1)/norm(e2)
```

0.0

```
>> acos(0.0)
```

```
>> dot(e1,e1)/norm(e1)/norm(e1)
```

```
>> acos(1)
```

```
>> u=[1 0]; v=[1 1]; dot(u,v)/norm(u)/norm(v)
```

```
>> acos(0.7071)/pi*180
```

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|}$$

```
>> dot(e1,e2)/norm(e1)/norm(e2)
```

0.0

```
>> acos(0.0)
```

1.5708

```
>> dot(e1,e1)/norm(e1)/norm(e1)
```

```
>> acos(1)
```

```
>> u=[1 0]; v=[1 1]; dot(u,v)/norm(u)/norm(v)
```

```
>> acos(0.7071)/pi*180
```

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|}$$

```
>> dot(e1,e2)/norm(e1)/norm(e2)
```

0.0

```
>> acos(0.0)
```

1.5708

```
>> dot(e1,e1)/norm(e1)/norm(e1)
```

1

```
>> acos(1)
```

```
>> u=[1 0]; v=[1 1]; dot(u,v)/norm(u)/norm(v)
```

```
>> acos(0.7071)/pi*180
```

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|}$$

```
>> dot(e1,e2)/norm(e1)/norm(e2)
```

0.0

```
>> acos(0.0)
```

1.5708

```
>> dot(e1,e1)/norm(e1)/norm(e1)
```

1

```
>> acos(1)
```

0.0

```
>> u=[1 0]; v=[1 1]; dot(u,v)/norm(u)/norm(v)
```

```
>> acos(0.7071)/pi*180
```

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|}$$

```
>> dot(e1,e2)/norm(e1)/norm(e2)
```

0.0

```
>> acos(0.0)
```

1.5708

```
>> dot(e1,e1)/norm(e1)/norm(e1)
```

1

```
>> acos(1)
```

0.0

```
>> u=[1 0]; v=[1 1]; dot(u,v)/norm(u)/norm(v)
```

0.7071

```
>> acos(0.7071)/pi*180
```

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|}$$

```
>> dot(e1,e2)/norm(e1)/norm(e2)
```

0.0

```
>> acos(0.0)
```

1.5708

```
>> dot(e1,e1)/norm(e1)/norm(e1)
```

1

```
>> acos(1)
```

0.0

```
>> u=[1 0]; v=[1 1]; dot(u,v)/norm(u)/norm(v)
```

0.7071

```
>> acos(0.7071)/pi*180
```

45.001