- New concepts:
  - Matrix addition and scaling
  - Matrix multiplication
  - Matrix addition properties
  - Matrix multiplication properties

Definition. *The sum of matrices* $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times n}$

$$\boldsymbol{A} = [a_{i,j}], \boldsymbol{B} = [b_{i,j}]$$

is the matrix $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$ *with components*

$$\boldsymbol{C} = [c_{i,j}], c_{i,j} = a_{i,j} + b_{i,j}$$

```
>> A=[1 0 1; 2 -1 3]; B=[2 1 -1; 1 1 -1]; C=A+B; disp(A)
```

```
>> disp(B)
```

```
>> A+B
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>> disp(B)
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```

```
   1   0   1
   2  -1   3
```

```
>> disp(B)
```

```
   2   1  -1
   1   1  -1
```

```
>> A+B
```

$$\begin{pmatrix} 3 & 1 & 0.0 \\ 3 & 0.0 & 2 \end{pmatrix}$$

Definition. *The scalar multiplication of matrix* $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ *by* $\alpha \in \mathbb{R}$ *is* $\boldsymbol{B} = \alpha\,\boldsymbol{A}$

$$\boldsymbol{A} = [a_{i,j}], \boldsymbol{B} = [b_{i,j}] = [\alpha\,a_{i,j}]$$

```
>> A=[1 0 1; 2 -1 3]; disp(A)
```

```
>> 2*A
```

```
>> A+A
```

```
>>
```

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>> A=[1 0 1; 2 -1 3]; disp(A)
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```
   1   0   1
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```

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>> 2*A
```

$$\begin{pmatrix} 2 & 0.0 & 2 \\ 4 & -2 & 6 \end{pmatrix}$$

```
>> A+A
```

```
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```

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```
>> A+A
```

$$\begin{pmatrix} 2 & 0.0 & 2 \\ 4 & -2 & 6 \end{pmatrix}$$

```
>>
```

Definition. *Consider matrices* $\boldsymbol{A} = [\ \boldsymbol{a}_1 \ \ \dots \ \ \boldsymbol{a}_n\ ] \in \mathbb{R}^{m \times n}$, *and* $\boldsymbol{X} = [\ \boldsymbol{x}_1 \ \ \dots \ \ \boldsymbol{x}_p\ ] \in \mathbb{R}^{n \times p}$. *The matrix product* $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{X}$ *is a matrix* $\boldsymbol{B} = [\ \boldsymbol{b}_1 \ \ \dots \ \ \boldsymbol{b}_p\ ] \in \mathbb{R}^{m \times p}$ *with column vectors given by the matrix vector products*

$$\boldsymbol{b}_k = \boldsymbol{A}\boldsymbol{x}_k, \text{for } k = 1, 2 \dots, p.$$

- A matrix-matrix product is simply a set of matrix-vector products, and hence expresses multiple linear combinations in a concise way.
- The dimensions of the matrices must be compatible, the number of rows of $\boldsymbol{X}$ must equal the number of columns of $\boldsymbol{A}$.
- A matrix-vector product is a special case of a matrix-matrix product when $p = 1$.
- We often write $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{X}$ in terms of columns as

$$[\ \boldsymbol{b}_1 \ \ \dots \ \ \boldsymbol{b}_p\ ] = \boldsymbol{A}[\ \boldsymbol{x}_1 \ \ \dots \ \ \boldsymbol{x}_p\ ] = [\ \boldsymbol{A}\boldsymbol{x}_1 \ \ \dots \ \ \boldsymbol{A}\boldsymbol{x}_p\ ]$$

```
>> A=[1 0 3; 2 1 4; -1 0 3]; disp(A)
```

```
>> X=[1 -1 0; 1 1 1; 0 1 0]; disp(X)
```

```
>> A*X
```

```
>> [A*X(:,1) A*X(:,2) A*X(:,3)]
```

```
>> A=[1 0 3; 2 1 4; -1 0 3]; disp(A)
```

```
    1   0   3
    2   1   4
   -1   0   3
```

```
>> X=[1 -1 0; 1 1 1; 0 1 0]; disp(X)
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>> [A*X(:,1) A*X(:,2) A*X(:,3)]
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>> A=[1 0 3; 2 1 4; -1 0 3]; disp(A)
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```
    1    0    3
    2    1    4
   -1    0    3
```

```
>> X=[1 -1 0; 1 1 1; 0 1 0]; disp(X)
```

```
    1   -1    0
    1    1    1
    0    1    0
```

```
>> A*X
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>> [A*X(:,1) A*X(:,2) A*X(:,3)]
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```
>> X=[1 -1 0; 1 1 1; 0 1 0]; disp(X)
```

```
   1  -1   0
   1   1   1
   0   1   0
```

```
>> A*X
```

$$\begin{pmatrix} 1 & 2 & 0.0 \\ 3 & 3 & 1 \\ -1 & 4 & 0.0 \end{pmatrix}$$

```
>> [A*X(:,1) A*X(:,2) A*X(:,3)]
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$$\begin{pmatrix} 1 & 2 & 0.0 \\ 3 & 3 & 1 \\ -1 & 4 & 0.0 \end{pmatrix}$$

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>> [A*X(:,1) A*X(:,2) A*X(:,3)]
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$$\begin{pmatrix} 1 & 2 & 0.0 \\ 3 & 3 & 1 \\ -1 & 4 & 0.0 \end{pmatrix}$$

Definition. *Consider matrices* $\boldsymbol{A} = [\ a_{i,j}\ ] \in \mathbb{R}^{m\times n}$, *and* $\boldsymbol{X} = [x_{i,j}] \in \mathbb{R}^{n\times p}$. *The matrix product* $\boldsymbol{B} = \boldsymbol{A}\,\boldsymbol{X} = [b_{i,j}]$ *is a matrix* $\boldsymbol{B} \in \mathbb{R}^{m\times p}$ *with components*

$$b_{i,j} = a_{i,1}\,x_{1,j} + a_{i,2}\,x_{2,j} + \cdots + a_{i,n}\,x_{n,j} = \sum_{k=1}^{n} a_{i,k}\,x_{k,j}$$

$$\boldsymbol{A}\,\boldsymbol{X} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & & a_{m,n} \end{bmatrix} \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,p} \\ x_{2,1} & x_{2,2} & \ldots & x_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & & x_{n,p} \end{bmatrix}$$

$$b_{2,1} = a_{2,1}\,x_{1,1} + a_{2,2}\,x_{2,1} + \cdots + a_{2,n}\,x_{n,1}$$

```
>> A=[1 0 3; 2 1 4; -1 0 3]; disp(A)
```

```
>> X=[1 -1 0; 1 1 1; 0 1 0]; disp(X)
```

```
>> A*X
```

```
>> dot(A(1,:),X(:,2))
```

```
>> dot(A(3,:),X(:,1))
```

```
>> A=[1 0 3; 2 1 4; -1 0 3]; disp(A)
```

```
    1    0    3
    2    1    4
   -1    0    3
```

```
>> X=[1 -1 0; 1 1 1; 0 1 0]; disp(X)
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>> A*X
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    1   0   3
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>> X=[1 -1 0; 1 1 1; 0 1 0]; disp(X)
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```
    1  -1   0
    1   1   1
    0   1   0
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```

Matrix multiplication example revisited

```
>> A=[1 0 3; 2 1 4; -1 0 3]; disp(A)
```

```
    1    0    3
    2    1    4
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```

```
>> X=[1 -1 0; 1 1 1; 0 1 0]; disp(X)
```

```
    1   -1    0
    1    1    1
    0    1    0
```

```
>> A*X
```

$$\begin{pmatrix} 1 & 2 & 0.0 \\ 3 & 3 & 1 \\ -1 & 4 & 0.0 \end{pmatrix}$$

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   1   0   3
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>> X=[1 -1 0; 1 1 1; 0 1 0]; disp(X)
```

```
   1  -1   0
   1   1   1
   0   1   0
```

```
>> A*X
```

$$\begin{pmatrix} 1 & 2 & 0.0 \\ 3 & 3 & 1 \\ -1 & 4 & 0.0 \end{pmatrix}$$

```
>> dot(A(1,:),X(:,2))
```

2

```
>> dot(A(3,:),X(:,1))
```

```
>> A=[1 0 3; 2 1 4; -1 0 3]; disp(A)
```

```
   1   0   3
   2   1   4
  -1   0   3
```

```
>> X=[1 -1 0; 1 1 1; 0 1 0]; disp(X)
```

```
   1  -1   0
   1   1   1
   0   1   0
```

```
>> A*X
```

$$\begin{pmatrix} 1 & 2 & 0.0 \\ 3 & 3 & 1 \\ -1 & 4 & 0.0 \end{pmatrix}$$

```
>> dot(A(1,:),X(:,2))
```

2

```
>> dot(A(3,:),X(:,1))
```

-1

- $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{B} \in \mathbb{R}^{m \times n}, \alpha, \beta, \gamma \in \mathbb{R}$
- Associativity: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$, $(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$
- Zero element: $\alpha + 0 = \alpha$, $\boldsymbol{A} + \mathbf{0} = \boldsymbol{A}$

$$\boldsymbol{A} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

- Opposite: $\alpha + (-\alpha) = \alpha + (-1)\alpha = 0$, $\boldsymbol{A} + (-\boldsymbol{A}) = \boldsymbol{A} + (-1)\boldsymbol{A} = \mathbf{0}$.
- Commutativity: $\alpha + \beta = \beta + \alpha$, $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$

- $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{B} \in \mathbb{R}^{n \times p}, \boldsymbol{C} \in \mathbb{R}^{p \times q}$, $\alpha, \beta, \gamma \in \mathbb{R}$
- Associativity: $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, $(\boldsymbol{A}\boldsymbol{B})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{B}\boldsymbol{C})$
- Unity element: $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$, $\boldsymbol{A}\boldsymbol{I} = \boldsymbol{I}\boldsymbol{A} = \boldsymbol{A}$ ($\boldsymbol{I}$ is the *identity matrix*)

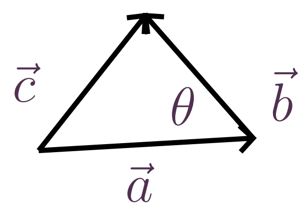
$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

- Inverse: $\alpha \neq 0$, $\alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = 1$, Would like $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$, but need to establish conditions for existence of $\boldsymbol{A}^{-1}$ (analogous to $\alpha \neq 0$)
- Commutativity: $\alpha\beta = \beta\alpha$. In general $\boldsymbol{A}\boldsymbol{B}$ need not be equal to $\boldsymbol{B}\boldsymbol{A}$.

- Dot product of $\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^m$: $\boldsymbol{u}\cdot\boldsymbol{v}=u_1v_1+\cdots+u_mv_m$ is a matrix multiplication

$$\boldsymbol{u}\cdot\boldsymbol{v}=\boldsymbol{u}^T\boldsymbol{v}=\begin{bmatrix} u_1 & \ldots & u_m \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}=\boldsymbol{v}^T\boldsymbol{u}=\begin{bmatrix} v_1 & \ldots & v_m \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

- The above form is useful in many cases, e.g., proving the cosine theorem



Cosine theorem: $\vec{c}=\vec{a}+\vec{b}$, $c^2=a^2+b^2-2ab\cos\theta$, with $a=\|\vec{a}\|$,$b=\|\vec{b}\|$,$c=\|\vec{c}\|$.

$\boldsymbol{c}=\boldsymbol{a}+\boldsymbol{b}\Rightarrow\boldsymbol{c}^T=\boldsymbol{a}^T+\boldsymbol{b}^T$, $\boldsymbol{c}^T\boldsymbol{c}=(\boldsymbol{a}^T+\boldsymbol{b}^T)(\boldsymbol{a}+\boldsymbol{b})\Rightarrow\|\boldsymbol{c}\|^2=\|\boldsymbol{a}\|^2+\|\boldsymbol{b}\|^2+2\boldsymbol{a}^T\boldsymbol{b}$

$\|\boldsymbol{c}\|^2=\|\boldsymbol{a}\|^2+\|\boldsymbol{b}\|^2+2\|\boldsymbol{a}\|\|\|\boldsymbol{b}\|\|\cos(\pi-\theta)\Rightarrow$

$$\|\boldsymbol{c}\|^2=\|\boldsymbol{a}\|^2+\|\boldsymbol{b}\|^2-2\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta\Rightarrow c^2=a^2+b^2-2ab\cos\theta.$$