- New concepts:
  - Matrix transpose
  - Block matrices
  - Block matrix addition, multiplication, transposition
  - Combined operations:
    - $\longrightarrow$ Addition and multiplication (Matrix distributivity)
    - $\longrightarrow$ Transposition and addition
    - $\longrightarrow$ Transposition and multiplication

Definition. *Given a matrix* $\boldsymbol{A} \in \mathbb{R}^{m \times n}$,

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the transpose of $\boldsymbol{A}$, *denoted as* $\boldsymbol{A}^T \in \mathbb{R}^{n \times m}$ *is*

$$\boldsymbol{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}.$$

$$[\boldsymbol{A}^T]_{i,j} = [\boldsymbol{A}]_{j,i}$$

Transposition switches row and column vectors

```
>> A=[1 2 3; 4 5 6; 7 8 9]; disp(A)
```

```
    1    2    3
    4    5    6
    7    8    9
```

```
>> transpose(A)
```

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

```
>> A'
```

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

```
>> u=[1 2 3]; u'
```

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

- $\boldsymbol{A} \in \mathbb{R}^{4\times 4}$ is a mtrix with 4 rows and 4 columns

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 2 \\ -1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 1 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

- It is useful to recognize common structures in a matrix

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{C} & \boldsymbol{B} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{F} \\ \boldsymbol{F} & \boldsymbol{E} \end{bmatrix}, \boldsymbol{E} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \boldsymbol{F} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Matrix block addition

$$\boldsymbol{A}=\begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{C} & \boldsymbol{B} \end{bmatrix}, \boldsymbol{D}=\begin{bmatrix} \boldsymbol{E} & \boldsymbol{F} \\ \boldsymbol{F} & \boldsymbol{E} \end{bmatrix},$$

$$\boldsymbol{A}+\boldsymbol{D}=\begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{C} & \boldsymbol{B} \end{bmatrix}+\begin{bmatrix} \boldsymbol{E} & \boldsymbol{F} \\ \boldsymbol{F} & \boldsymbol{E} \end{bmatrix}=\begin{bmatrix} \boldsymbol{B}+\boldsymbol{E} & \boldsymbol{C}+\boldsymbol{F} \\ \boldsymbol{C}+\boldsymbol{F} & \boldsymbol{B}+\boldsymbol{E} \end{bmatrix}$$

- Matrix block multiplication

$$\boldsymbol{A}\boldsymbol{D}=\begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{C} & \boldsymbol{B} \end{bmatrix}\begin{bmatrix} \boldsymbol{E} & \boldsymbol{F} \\ \boldsymbol{F} & \boldsymbol{E} \end{bmatrix}=\begin{bmatrix} \boldsymbol{B}\boldsymbol{E}+\boldsymbol{C}\boldsymbol{F} & \boldsymbol{B}\boldsymbol{F}+\boldsymbol{C}\boldsymbol{E} \\ \boldsymbol{C}\boldsymbol{E}+\boldsymbol{B}\boldsymbol{F} & \boldsymbol{C}\boldsymbol{F}+\boldsymbol{B}\boldsymbol{E} \end{bmatrix}$$

- Matrix block transposition

$$\boldsymbol{M}=\begin{bmatrix} \boldsymbol{U} & \boldsymbol{V} \\ \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix}, \boldsymbol{M}^T=\begin{bmatrix} \boldsymbol{U}^T & \boldsymbol{X}^T \\ \boldsymbol{V}^T & \boldsymbol{Y}^T \end{bmatrix}$$

- $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ contains n column vectors with m components each

$$\boldsymbol{A} = [\ \boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \dots \quad \boldsymbol{a}_n\ ]$$

- The transpose switches rows and columns

$$\boldsymbol{A}^T = \begin{bmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \vdots \\ \boldsymbol{a}_n^T \end{bmatrix} \in \mathbb{R}^{n \times m}$$

has n rows and m columns

```
>> A=[1 2 -1 0; 2 1 0 2; -1 0 1 2; 0 2 2 1]; disp(A)

   1   2  -1   0
   2   1   0   2
  -1   0   1   2
   0   2   2   1
```

```
>> B=A(1:2,1:2); C=A(1:2,3:4); [B C; C B]
```

$$\begin{pmatrix} 1 & 2 & -1 & 0.0 \\ 2 & 1 & 0.0 & 2 \\ -1 & 0.0 & 1 & 2 \\ 0.0 & 2 & 2 & 1 \end{pmatrix}$$

```
>> D=[1 -1 1 0; -1 1 0 1; 1 0 1 -1; 0 1 -1 1];
>> E=D(1:2,1:2); F=D(1:2,3:4); [D [E F; F E]]
```

$$\begin{pmatrix} 1 & -1 & 1 & 0.0 & 1 & -1 & 1 & 0.0 \\ -1 & 1 & 0.0 & 1 & -1 & 1 & 0.0 & 1 \\ 1 & 0.0 & 1 & -1 & 1 & 0.0 & 1 & -1 \\ 0.0 & 1 & -1 & 1 & 0.0 & 1 & -1 & 1 \end{pmatrix}$$

Block operation examples: addition

$$A+D=\begin{bmatrix} B & C \\ C & B \end{bmatrix}+\begin{bmatrix} E & F \\ F & E \end{bmatrix}=\begin{bmatrix} B+E & C+F \\ C+F & B+E \end{bmatrix}$$

```
>> A+D
```

$$\begin{pmatrix} 2 & 1 & 0.0 & 0.0 \\ 1 & 2 & 0.0 & 3 \\ 0.0 & 0.0 & 2 & 1 \\ 0.0 & 3 & 1 & 2 \end{pmatrix}$$

```
>> [B C; C B] + [E F; F E]
```

$$\begin{pmatrix} 2 & 1 & 0.0 & 0.0 \\ 1 & 2 & 0.0 & 3 \\ 0.0 & 0.0 & 2 & 1 \\ 0.0 & 3 & 1 & 2 \end{pmatrix}$$

```
>> [B+E C+F; C+F B+E]
```

$$\begin{pmatrix} 2 & 1 & 0.0 & 0.0 \\ 1 & 2 & 0.0 & 3 \\ 0.0 & 0.0 & 2 & 1 \\ 0.0 & 3 & 1 & 2 \end{pmatrix}$$

Block operation examples: multiplication

$$AD = \begin{bmatrix} B & C \\ C & B \end{bmatrix} \begin{bmatrix} E & F \\ F & E \end{bmatrix} = \begin{bmatrix} BE+CF & BF+CE \\ CE+BF & CF+BE \end{bmatrix}$$

```
>> A*D
```

$$\begin{pmatrix} -2 & 1 & 0.0 & 3 \\ 1 & 1 & 0.0 & 3 \\ 0.0 & 3 & -2 & 1 \\ 0.0 & 3 & 1 & 1 \end{pmatrix}$$

```
>> [B*E+C*F B*F+C*E; C*E+B*F C*F+B*E]
```

$$\begin{pmatrix} -2 & 1 & 0.0 & 3 \\ 1 & 1 & 0.0 & 3 \\ 0.0 & 3 & -2 & 1 \\ 0.0 & 3 & 1 & 1 \end{pmatrix}$$

```
>>
```

$$M = \begin{bmatrix} U & V \\ X & Y \end{bmatrix}, M^T = \begin{bmatrix} U^T & X^T \\ V^T & Y^T \end{bmatrix}$$

```
>> M=[1 2 3 4; 5 6 7 8; -1 -2 -3 -4; -5 -6 -7 -8]; disp([M M'])
```

```
   1   2   3   4   1   5  -1  -5
   5   6   7   8   2   6  -2  -6
  -1  -2  -3  -4   3   7  -3  -7
  -5  -6  -7  -8   4   8  -4  -8
```

```
>> U=M(1:2,1:2); V=M(1:2,3:4); X=M(3:4,1:2); Y=M(3:4,3:4);
```

```
>> [U'␣X'; V'␣Y']
```

$$\begin{pmatrix} 1 & 5 & -1 & -5 \\ 2 & 6 & -2 & -6 \\ 3 & 7 & -3 & -7 \\ 4 & 8 & -4 & -8 \end{pmatrix}$$

```
>>
```

- Distributivity $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{n \times p}$, $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}\boldsymbol{B} + \boldsymbol{A}\boldsymbol{C}$
- Transposition of sum $(\boldsymbol{B}+\boldsymbol{C})^T = \boldsymbol{B}^T + \boldsymbol{C}^T$
- Transposition of scaling $\alpha \in \mathbb{R}$, $(\alpha \boldsymbol{A})^T = \alpha \boldsymbol{A}^T$
- Transposition of product: Start by matrix-vector product for $\boldsymbol{x} \in \mathbb{R}^{n \times 1}$

$$\boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} \boldsymbol{a}_1 & \dots & \boldsymbol{a}_n \end{pmatrix} \boldsymbol{x} = x_1 \boldsymbol{a}_1 + \ldots + x_n \boldsymbol{a}_n \Rightarrow$$

$$(\boldsymbol{A}\boldsymbol{x})^T = x_1 \boldsymbol{a}_1^T + \ldots + x_n \boldsymbol{a}_n^T = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_1^T \\ \vdots \\ \boldsymbol{a}_n^T \end{bmatrix} = \boldsymbol{x}^T \boldsymbol{A}^T$$

- Transposition of product $(\boldsymbol{A}\boldsymbol{B})^T = \boldsymbol{B}^T \boldsymbol{A}^T$

$$(\boldsymbol{A}\boldsymbol{B})^T = \begin{pmatrix} \boldsymbol{A}\boldsymbol{b}_1 & \dots & \boldsymbol{A}\boldsymbol{b}_n \end{pmatrix}^T = \begin{pmatrix} (\boldsymbol{A}\boldsymbol{b}_1)^T \\ \vdots \\ (\boldsymbol{A}\boldsymbol{b}_n)^T \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1^T \boldsymbol{A}^T \\ \vdots \\ \boldsymbol{b}_n^T \boldsymbol{A}^T \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1^T \\ \vdots \\ \boldsymbol{b}_n^T \end{pmatrix} \boldsymbol{A}^T$$