- New concepts:
  - Linear transformation
  - Matrix of a linear transformation
  - Common transformations: stretching, orthogonal projection, reflection, rotation
  - Composition of linear transformations

- Calculus studies $f: \mathbb{R} \to \mathbb{R}$, functions defined on reals with values in reals
- Linear algebra studies $T: \mathbb{R}^n \to \mathbb{R}^m$, mapping of vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$
- Of special interest: mappings that preserve linear combinations

$$T(\alpha \boldsymbol{u} + \beta \boldsymbol{v}) = \alpha T(\boldsymbol{u}) + \beta T(\boldsymbol{v})$$

  Such mappings are said to be *linear*.
- Examples, counter-examples:
  - $f: \mathbb{R} \to \mathbb{R}$, $m = n = 1$, $f(x) = a x$ is a linear mapping

$$f(\alpha x + \beta y) = a(\alpha x + \beta y) = \alpha a x + \alpha \beta y = \alpha f(x) + \beta f(y)$$

  - $g: \mathbb{R} \to \mathbb{R}$, $m = n = 1$, $g(x) = a x + b$ is *not* a linear mapping

$$g(\alpha x + \beta y) = a(\alpha x + \beta y) + b = \alpha a x + b + \alpha \beta y = \alpha g(x) + \beta g(y) - b$$

  - Matrix multiplication is linear $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$

$$T(\alpha \boldsymbol{x} + \beta \boldsymbol{y}) = \boldsymbol{A}(\alpha \boldsymbol{x} + \beta \boldsymbol{y}) = \alpha \boldsymbol{A}\boldsymbol{x} + \beta \boldsymbol{A}\boldsymbol{y} = \alpha T(\boldsymbol{x}) + \beta T(\boldsymbol{y})$$

  - $S: \mathbb{R}^n \to \mathbb{R}^m$, $S(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ is *not* a linear mapping

- $\boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{v} = \boldsymbol{I}\boldsymbol{v}$, $\boldsymbol{I} = [\ \boldsymbol{e}_1\ \ \boldsymbol{e}_2\ \ \dots\ \ \boldsymbol{e}_n\ ]$

$$\boldsymbol{v} = [\ \boldsymbol{e}_1\ \ \boldsymbol{e}_2\ \ \dots\ \ \boldsymbol{e}_n\ ] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\boldsymbol{e}_1 + \cdots + v_n\boldsymbol{e}_n = v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- $T: \mathbb{R}^n \to \mathbb{R}^m$ a linear mapping (transformation)

$$T(\boldsymbol{v}) = T(v_1\boldsymbol{e}_1 + \cdots + v_n\boldsymbol{e}_n) = v_1 T(\boldsymbol{e}_1) + \cdots + v_n T(\boldsymbol{e}_n)$$

- Let $\boldsymbol{A} = [\ T(\boldsymbol{e}_1)\ \ T(\boldsymbol{e}_2)\ \ \dots\ \ T(\boldsymbol{e}_n)\ ] \in \mathbb{R}^{m \times n}$, $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$. $\boldsymbol{A}$ is the *standard matrix* of a linear transformation

- T stretches the x component of $\boldsymbol{v} \in \mathbb{R}^2$ by factor α, the y component by β

$$T(\boldsymbol{e}_1) = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, T(\boldsymbol{e}_2) = \begin{bmatrix} 0 \\ \beta \end{bmatrix}, \boldsymbol{A} = [\ T(\boldsymbol{e}_1)\ \ T(\boldsymbol{e}_2)\ ] = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2\alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- In general a diagonal matrix describes stretching with factors $\lambda_1, \lambda_2, \ldots, \lambda_m > 0$

$$\boldsymbol{A} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_m \end{bmatrix}$$

- Orthogonal projection of $\boldsymbol{v} \in \mathbb{R}^m$ along direction $\boldsymbol{u} \in \mathbb{R}^m$, $\|\boldsymbol{u}\| = 1$

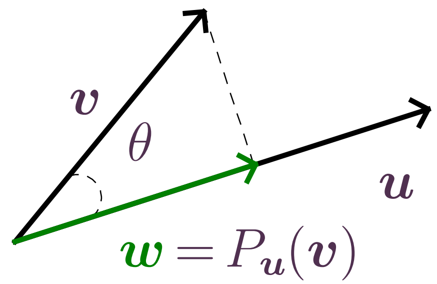

Figure 1. Orthogonal projection operation $P_{\boldsymbol{u}}$.

- $\boldsymbol{w} = (\|\boldsymbol{v}\| \cos\theta)\, \boldsymbol{u} = \left(\|\boldsymbol{v}\| \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\, \|\boldsymbol{v}\|} \right) \boldsymbol{u} = (\boldsymbol{u}^T \boldsymbol{v}) \boldsymbol{u} = \boldsymbol{u} (\boldsymbol{u}^T \boldsymbol{v}) = (\boldsymbol{u}\, \boldsymbol{u}^T) \boldsymbol{v} \Rightarrow$
- Projection matrix $\boldsymbol{P}_{\boldsymbol{u}} = \boldsymbol{u}\, \boldsymbol{u}^T$ ($\|\boldsymbol{u}\| = 1$)

- Projection along $\boldsymbol{e}_1$ direction in $\mathbb{R}^2$

$$\boldsymbol{P}_{\boldsymbol{e}_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{P}_{\boldsymbol{e}_1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

- Projection along direction of vector $\boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$:
  - First obtain a vector of unit norm $\boldsymbol{u} = \boldsymbol{w} / \|\boldsymbol{w}\| = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
  - Projection matrix

$$\boldsymbol{P}_{\boldsymbol{u}} = \boldsymbol{u}\boldsymbol{u}^T = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

- Reflection of $\boldsymbol{u} \in \mathbb{R}^2$ across x_1 axis

$$T(\boldsymbol{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, T(\boldsymbol{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \boldsymbol{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}.$$

- Reflection of $\boldsymbol{u} \in \mathbb{R}^2$ across $\boldsymbol{w} \in \mathbb{R}^2$, $\|\boldsymbol{w}\| = 1$, $\boldsymbol{v} = T_{\boldsymbol{w}}(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$. Two steps:
  1. Project $\boldsymbol{u}$ onto direction of $\boldsymbol{w}$, $\boldsymbol{y} = \boldsymbol{w}\boldsymbol{w}^T\boldsymbol{u}$, $\boldsymbol{y} = \boldsymbol{u} + \boldsymbol{z}$.
  2. Go from $\boldsymbol{u}$ twice the vector $\boldsymbol{z}$ to obtain the reflection

$$\boldsymbol{v} = \boldsymbol{A}\boldsymbol{u} = \boldsymbol{u} + 2\boldsymbol{z} = \boldsymbol{u} + 2(\boldsymbol{y} - \boldsymbol{u}) = -\boldsymbol{u} + 2\boldsymbol{w}\boldsymbol{w}^T\boldsymbol{u} = (2\boldsymbol{w}\boldsymbol{w}^T - \boldsymbol{I})\boldsymbol{u}$$

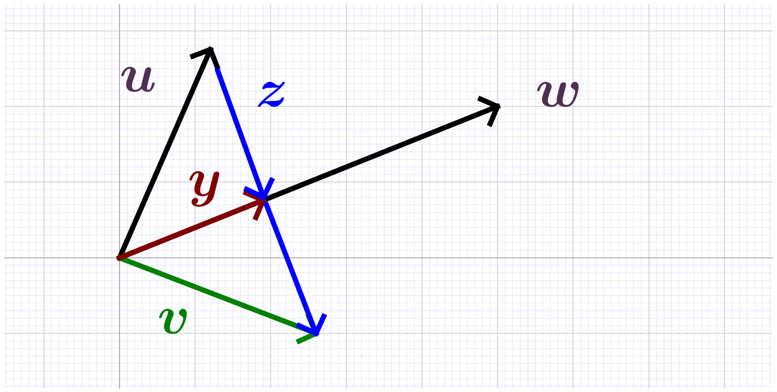

Figure 2. Reflection matrix diagram $\boldsymbol{A} = 2\boldsymbol{w}\boldsymbol{w}^T - \boldsymbol{I}$.

- Reflect $\boldsymbol{e}_2$ across $\boldsymbol{w} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\boldsymbol{A} = 2\boldsymbol{w}\boldsymbol{w}^T - \boldsymbol{I} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\boldsymbol{A}\boldsymbol{e}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Construct standard rotation matrix by rotating $\boldsymbol{e}_1, \boldsymbol{e}_2$

$$\boldsymbol{A} = [\ T(\boldsymbol{e}_1)\ \ T(\boldsymbol{e}_2)\ ] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

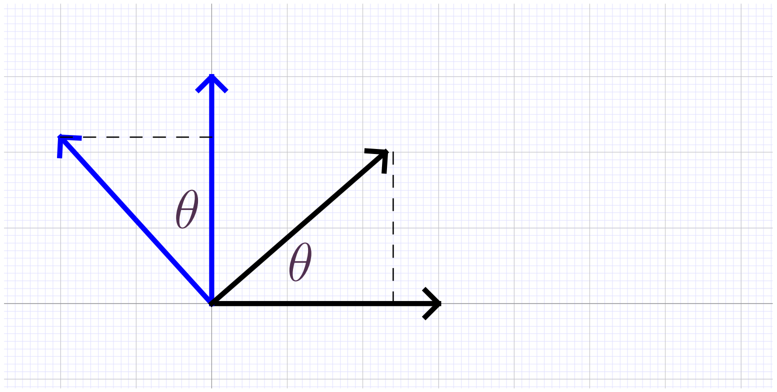

Figure 3. Diagram for construction of rotation matrix

- Consider two transformations $S(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$, $T(\boldsymbol{u}) = \boldsymbol{B}\boldsymbol{u}$. Composition $S(T(\boldsymbol{u}))$

$$R(\boldsymbol{u}) = S(T(\boldsymbol{u})) = S(\boldsymbol{B}\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{B}\boldsymbol{u} = \boldsymbol{C}\boldsymbol{u}$$

- Matrix of transformation composition is product of the individual transformation matrices
- Example: Rotation by φ followed by rotation by θ in $\mathbb{R}^2$

$$\begin{bmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

$$\cos(\theta+\varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)$$

$$\sin(\theta+\varphi) = \sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi)$$