- New concepts:
  - Bordered matrix for a linear system
  - Similarity transformations of a linear system
  - Gaussian elimination

- Idea: make one fewer unknown appear in each equation. Use first equation to eliminate x_1 in equations 2,3

$$\left\{\begin{array}{rcl} x_1+2\,x_2-x_3 & = & 2\\ 2\,x_1-x_2+x_3 & = & 2\\ 3\,x_1-x_2-x_3 & = & 1 \end{array}\right. \Rightarrow \left\{\begin{array}{rcr} x_1+2\,x_2-x_3 & = & 2\\ -5\,x_2+3\,x_3 & = & -2\\ -7\,x_2+2\,x_3 & = & -5 \end{array}\right.$$

- Use second equation to eliminate x_2 in equation 3

$$\left\{\begin{array}{rcr} x_1+2\,x_2-x_3 & = & 2\\ -5\,x_2+3\,x_3 & = & -2\\ -7\,x_2+2\,x_3 & = & -5 \end{array}\right. \Rightarrow \left\{\begin{array}{rcr} x_1+2\,x_2-x_3 & = & 2\\ -5\,x_2+3\,x_3 & = & -2\\ -\dfrac{11}{5}\,x_3 & = & -\dfrac{11}{5} \end{array}\right.$$

- Start finding components from last to first to obtain $x_3=1$, $x_2=1$,$x_1=1$

- Explicitly writing the unknowns x_1, x_2, x_3 is not necessary. Intoduce the "bordered" matrix

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 1 & 2 \\ 3 & -1 & -1 & 1 \end{bmatrix}$$

- Define allowed operations:
  - multiply a row by a non-zero scalar
  - add a row to another
- Bordered matrices obtained by the allowed operations are said to be *similar*, in that the solution of the linear system stays the same

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 1 & 2 \\ 3 & -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & -7 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & -\frac{11}{5} & -\frac{11}{5} \end{bmatrix}$$

- To find solution, use allowed operations to make an identity matrix appear

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & -\frac{11}{5} & -\frac{11}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

- The above constitute "Gaussian elimination"

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 1 & 2 \\ 3 & -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & -7 & 2 & -5 \end{bmatrix}$$

```
>> A=[1 2 -1 2; 2 -1 1 2; 3 -1 -1 1]; A(2,:)=A(2,:)-2*A(1,:);
   A(3,:)=A(3,:)-3*A(1,:); disp(A)
```

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- The above constitute "Gaussian elimination"

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 1 & 2 \\ 3 & -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & -7 & 2 & -5 \end{bmatrix}$$

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```

```
   1   2  -1   2
   0  -5   3  -2
   0  -7   2  -5
```

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \in \mathbb{R}^{3\times 3}, \boldsymbol{b} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \in \mathbb{R}^3, \boldsymbol{c} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \in \mathbb{R}^3 \tag{1}$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 & = 3 \\ x_2 + x_3 & = 1 \\ x_1 + 2x_2 + 3x_3 & = 3 \end{cases} \Leftrightarrow \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \begin{cases} y_1 + 2y_2 + 3y_3 & = 3 \\ y_2 + y_3 & = 1 \\ y_1 + 2y_2 + 3y_3 & = 4 \end{cases} \Leftrightarrow \boldsymbol{A}\boldsymbol{y} = \boldsymbol{c}$$

Form the bordered matrix in both cases, and reduce to triangular form

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow \begin{cases} x_1 + 2x_2 + 3x_3 & = 3 \\ x_2 + x_3 & = 1 \text{ Infinite solutions} \\ 0 & = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \Leftrightarrow \begin{cases} x_1 + 2x_2 + 3x_3 & = 3 \\ x_2 + x_3 & = 1 \text{ No solutions} \\ 0 & = 1 \end{cases}$$