- New concepts:
  - Row echelon form
  - Elementary matrices
  - Matrix inverse

- Use similarity transformations to *reduced row echelon form*:
  - All zero rows are below non-zero rows
  - First non-zero entry on a row is called the *leading entry*
  - In each non-zero row, the leading entry is to the left of lower leading entries
  - Each leading entry equals 1 and is the only non-zero entry in its column
- *Row echelon form*:
  - Allow additional non-zero elements in a column, above the leading entry

```
>> A=[1 2 3; 0 1 1; 1 2 3]; rref(A)
```

```
>>
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```
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```

$$\begin{pmatrix} 1 & 0.0 & 1 \\ 0.0 & 1 & 1 \\ 0.0 & 0.0 & 0.0 \end{pmatrix}$$

```
>>
```

After carrying out rref on bordered matrix $[\ \boldsymbol{A} \mid \boldsymbol{b}\ ]$, if:

- there is a row with $[\ 0\ \ 0\ \ \ldots\ \ 0 \mid 1\ ] \Rightarrow$ No solutions
- the result is of form $[\ \boldsymbol{I} \mid \boldsymbol{c}\ ] \Rightarrow$ Unique solution
- there is no row of form $[\ 0\ \ 0\ \ \ldots\ \ 0 \mid 1\ ]$, and there is a row of all zeros $[\ 0\ \ 0\ \ \ldots\ \ 0 \mid 0\ ] \Rightarrow$ Infinitely many solutions

Examples

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \text{Infinitely many solutions}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 8 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \text{Unique solution}$$

- Denote a permutation by

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & m \\ i_1 & i_2 & \cdots & i_m \end{pmatrix}$$

  with $i_1, \ldots, i_m \in \{1, \ldots, m\}$, $i_j \neq i_k$ for $j \neq k$

- The sign of a permutation, $\nu(\sigma)$ is the number of pair swaps needed to obtain the permutation starting from the identity permutation

$$\begin{pmatrix} 1 & 2 & \ldots & m \\ 1 & 2 & \cdots & m \end{pmatrix}$$

- A permutation can be specified by a permutation matrix $\boldsymbol{P}$ obtained from $\boldsymbol{I}$ by swapping rows and columns $k \leftrightarrow i_k$

- Recall the basic operation in row echelon reduction: constructing a linear combination of rows to form zeros beneath the main diagonal, e.g.

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix} \sim \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & \dots & a_{2m} - \frac{a_{21}}{a_{11}} a_{1m} \\ 0 & a_{32} - \frac{a_{31}}{a_{11}} a_{12} & \dots & a_{3m} - \frac{a_{31}}{a_{11}} a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} - \frac{a_{m1}}{a_{11}} a_{12} & \dots & a_{mm} - \frac{a_{m1}}{a_{11}} a_{1m} \end{pmatrix}$$

- This can be stated as a matrix multiplication operation, with $l_{i1} = a_{i1} / a_{11}$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -l_{21} & 1 & 0 & \dots & 0 \\ -l_{31} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -l_{m1} & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22} - l_{21} a_{12} & \dots & a_{2m} - l_{21} a_{1m} \\ 0 & a_{32} - l_{31} a_{12} & \dots & a_{3m} - l_{31} a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} - l_{m1} a_{12} & \dots & a_{mm} - l_{m1} a_{1m} \end{pmatrix}$$

Definition. *The matrix*

$$\boldsymbol{L}_k = \begin{pmatrix} 1 & \dots & 0 & \dots & 1 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & -l_{k+1,k} & \dots & 0 \\ 0 & \dots & -l_{k+2,k} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & -l_{m,k} & \dots & 1 \end{pmatrix}$$

with $l_{i,k} = a_{i,k}^{(k)} / a_{k,k}^{(k)}$, *and* $\boldsymbol{A}^{(k)} = (a_{i,j}^{(k)})$ *the matrix obtained after step* k *of row echelon reduction (or, equivalently, Gaussian elimination) is called a Gaussian multiplier matrix.*

Permutation and Gaussian multiplier matrices are *elementary matrices*.

- Consider elementary matrices

$$\boldsymbol{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \boldsymbol{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \boldsymbol{E}_1\boldsymbol{E}_2 = \boldsymbol{E}_2\boldsymbol{E}_1 = \boldsymbol{I},$$

  stating that $\boldsymbol{E}_1$ undoes the effect of $\boldsymbol{E}_2$.
- $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ is invertible if there exists $\boldsymbol{X} \in \mathbb{R}^{m\times m}$ such that

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{A} = \boldsymbol{I}$$

- Notation $\boldsymbol{X} = \boldsymbol{A}^{-1}$, is the *inverse* of $\boldsymbol{A}$.