- New concepts:
  - Gauss-Jordan algorithm
  - Existence of matrix inverse
  - Operations with inverses

- Recall that the Gaussian multiplier matrix ...

$$\boldsymbol{L}_k = \begin{pmatrix} 1 & \dots & 0 & \dots & 1 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & -l_{k+1,k} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & -l_{m,k} & \dots & 1 \end{pmatrix}$$

- ... has inverse (matrix that "undoes" the linear transformation)

$$\boldsymbol{L}_k^{-1} = \begin{pmatrix} 1 & \dots & 0 & \dots & 1 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & l_{k+1,k} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & l_{m,k} & \dots & 1 \end{pmatrix}$$

- What about general square matrices $\boldsymbol{A} \in \mathbb{R}^{m \times m}$? How to find inverse
- $\boldsymbol{X}$ is inverse if $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{I}$ or

$$\boldsymbol{A}[\ \boldsymbol{x}_1\ \ \boldsymbol{x}_2\ \ \ldots\ \ \boldsymbol{x}_m\ ] = [\ \boldsymbol{A}\boldsymbol{x}_1\ \ \boldsymbol{A}\boldsymbol{x}_2\ \ \ldots\ \ \boldsymbol{A}\boldsymbol{x}_m\ ] = [\ \boldsymbol{e}_1\ \ \boldsymbol{e}_2\ \ \ldots\ \ \boldsymbol{e}_m\ ]$$

- Find the inverse is equivalent to solving systems $\boldsymbol{A}\boldsymbol{x}_1 = \boldsymbol{e}_1, \ldots, \boldsymbol{A}\boldsymbol{x}_m = \boldsymbol{e}_m$
- Gauss Jordan algoritm generalizes Gaussian elimination that solves a single linear system to solving m systems simultaneously by forming the bordered matrix $[\ \boldsymbol{A}\ \mid\ \boldsymbol{I}\ ]$

$$[\ \boldsymbol{A}\ \mid\ \boldsymbol{I}\ ] \sim [\ \boldsymbol{I}\ \mid\ \boldsymbol{X}\ ]$$

- Example

```
>> A=[1 2 1; -1 0 2; 2 -1 -4]; AX=[A eye(3)]; format rat; disp(AX);
```

```
>>
```

- What about general square matrices $\boldsymbol{A} \in \mathbb{R}^{m \times m}$? How to find inverse
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- Find the inverse is equivalent to solving systems $\boldsymbol{A}\boldsymbol{x}_1 = \boldsymbol{e}_1, \ldots, \boldsymbol{A}\boldsymbol{x}_m = \boldsymbol{e}_m$
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- Example

```
>> A=[1 2 1; -1 0 2; 2 -1 -4]; AX=[A eye(3)]; format rat; disp(AX);
```

```
        1           2           1           1           0           0
       -1           0           2           0           1           0
        2          -1          -4           0           0           1
```

```
>>
```

```
>> AX(2,:)=AX(2,:)+AX(1,:); AX(3,:)=AX(3,:)-2*AX(1,:); disp(AX);
```

```
>> AX(2,:)=(1/2)*AX(2,:); disp(AX);
```

```
>> AX(3,:)=AX(3,:)+5*AX(2,:); disp(AX);
```

```
>> AX(3,:)=(2/3)*AX(3,:); disp(AX);
```

```
>> AX(2,:)=AX(2,:)+AX(1,:); AX(3,:)=AX(3,:)-2*AX(1,:); disp(AX);
```

```
     1     2     1     1     0     0
     0     2     3     1     1     0
     0    -5    -6    -2     0     1
```

```
>> AX(2,:)=(1/2)*AX(2,:); disp(AX);
```

```
>> AX(3,:)=AX(3,:)+5*AX(2,:); disp(AX);
```

```
>> AX(3,:)=(2/3)*AX(3,:); disp(AX);
```

Gauss-Jordan example (cont)

```
>> AX(2,:)=AX(2,:)+AX(1,:); AX(3,:)=AX(3,:)-2*AX(1,:); disp(AX);
```

```
         1         2         1         1         0         0
         0         2         3         1         1         0
         0        -5        -6        -2         0         1
```

```
>> AX(2,:)=(1/2)*AX(2,:); disp(AX);
```

```
         1         2         1         1         0         0
         0         1       3/2       1/2       1/2         0
         0        -5        -6        -2         0         1
```

```
>> AX(3,:)=AX(3,:)+5*AX(2,:); disp(AX);
```

```
>> AX(3,:)=(2/3)*AX(3,:); disp(AX);
```

```
>> AX(2,:)=AX(2,:)+AX(1,:); AX(3,:)=AX(3,:)-2*AX(1,:); disp(AX);
```

```
     1     2     1     1     0     0
     0     2     3     1     1     0
     0    -5    -6    -2     0     1
```

```
>> AX(2,:)=(1/2)*AX(2,:); disp(AX);
```

```
     1     2     1     1     0     0
     0     1   3/2   1/2   1/2     0
     0    -5    -6    -2     0     1
```

```
>> AX(3,:)=AX(3,:)+5*AX(2,:); disp(AX);
```

```
     1     2     1     1     0     0
     0     1   3/2   1/2   1/2     0
     0     0   3/2   1/2   5/2     1
```

```
>> AX(3,:)=(2/3)*AX(3,:); disp(AX);
```

```
>> AX(2,:)=AX(2,:)+AX(1,:); AX(3,:)=AX(3,:)-2*AX(1,:); disp(AX);
```

```
     1     2     1     1     0     0
     0     2     3     1     1     0
     0    -5    -6    -2     0     1
```

```
>> AX(2,:)=(1/2)*AX(2,:); disp(AX);
```

```
     1     2     1     1     0     0
     0     1   3/2   1/2   1/2     0
     0    -5    -6    -2     0     1
```

```
>> AX(3,:)=AX(3,:)+5*AX(2,:); disp(AX);
```

```
     1     2     1     1     0     0
     0     1   3/2   1/2   1/2     0
     0     0   3/2   1/2   5/2     1
```

```
>> AX(3,:)=(2/3)*AX(3,:); disp(AX);
```

```
     1     2     1     1     0     0
     0     1   3/2   1/2   1/2     0
     0     0     1   1/3   5/3   2/3
```

```
>> AX(2,:)=AX(2,:)-(3/2)*AX(3,:); AX(1,:)=AX(1,:)-AX(3,:); disp(AX);
```

```
>> AX(1,:)=AX(1,:)-2*AX(2,:); X=AX(:,4:6); disp(AX);
```

```
>> format short; disp([A*X X*A]);
```

```
>>
```

```
>> AX(2,:)=AX(2,:)-(3/2)*AX(3,:); AX(1,:)=AX(1,:)-AX(3,:); disp(AX);
```

```
       1         2         0       2/3      -5/3      -2/3
       0         1         0         0        -2        -1
       0         0         1       1/3       5/3       2/3
```

```
>> AX(1,:)=AX(1,:)-2*AX(2,:); X=AX(:,4:6); disp(AX);
```

```
>> format short; disp([A*X X*A]);
```

```
>>
```

```
>> AX(2,:)=AX(2,:)-(3/2)*AX(3,:); AX(1,:)=AX(1,:)-AX(3,:); disp(AX);

          1          2          0        2/3       -5/3       -2/3
          0          1          0          0         -2         -1
          0          0          1        1/3        5/3        2/3

>> AX(1,:)=AX(1,:)-2*AX(2,:); X=AX(:,4:6); disp(AX);

          1          0          0        2/3        7/3        4/3
          0          1          0          0         -2         -1
          0          0          1        1/3        5/3        2/3

>> format short; disp([A*X X*A]);
>>
```

```
>> AX(2,:)=AX(2,:)-(3/2)*AX(3,:); AX(1,:)=AX(1,:)-AX(3,:); disp(AX);
```

```
     1     2     0   2/3  -5/3  -2/3
     0     1     0     0    -2    -1
     0     0     1   1/3   5/3   2/3
```

```
>> AX(1,:)=AX(1,:)-2*AX(2,:); X=AX(:,4:6); disp(AX);
```

```
     1     0     0   2/3   7/3   4/3
     0     1     0     0    -2    -1
     0     0     1   1/3   5/3   2/3
```

```
>> format short; disp([A*X X*A]);
```

```
    1.0000         0    0.0000    1.0000         0         0
   -0.0000    1.0000   -0.0000         0    1.0000         0
    0.0000    0.0000    1.0000         0         0    1.0000
```

```
>>
```

- When does a matrix inverse exist? $\boldsymbol{A} \in \mathbb{R}^{m \times m}$
  - a $\boldsymbol{A}$ invertible
  - b $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for all $\boldsymbol{b} \in \mathbb{R}^m$
  - c $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ has a unique solution
  - d The reduced row echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}$
  - e $\boldsymbol{A}$ can be written as product of elementary matrices

$$a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow e \Rightarrow a$$

$a \Rightarrow b$ $\boldsymbol{A}$ invertible $\Rightarrow$ $\boldsymbol{A}^{-1}$ exists, and $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$ is a solution $\boldsymbol{A}(\boldsymbol{A}^{-1}\boldsymbol{b}) = (\boldsymbol{A}\boldsymbol{A}^{-1})\boldsymbol{b} = \boldsymbol{b}$. If there were two solutions $\boldsymbol{x}, \boldsymbol{y}$, then

$$\boldsymbol{x} - \boldsymbol{y} = (\boldsymbol{A}^{-1}\boldsymbol{A})(\boldsymbol{x} - \boldsymbol{y}) = \boldsymbol{A}^{-1}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{A}\boldsymbol{y}) = \boldsymbol{A}^{-1}(\boldsymbol{b} - \boldsymbol{b}) = \boldsymbol{A}^{-1}\mathbf{0} = \mathbf{0}.$$

$b \Rightarrow c$ Choose $\boldsymbol{b} = \mathbf{0}$

$c \Rightarrow d$ $[\,\boldsymbol{A} \mid \mathbf{0}\,] \sim [\,\boldsymbol{U} \mid \mathbf{0}\,]$. If $\boldsymbol{U} \neq \boldsymbol{I}$ there is a row of zeros, and solution is not unique. If solution is unique then $\boldsymbol{U} = \boldsymbol{I}$

$d \Rightarrow e$ $[\,\boldsymbol{A} \mid \mathbf{0}\,] \sim [\,\boldsymbol{I} \mid \mathbf{0}\,]$ implies $\boldsymbol{E}_k \ldots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I} \Rightarrow \boldsymbol{A} = \boldsymbol{E}_1^{-1} \ldots \boldsymbol{E}_k^{-1}$

$e \Rightarrow a$ $\boldsymbol{A} = \boldsymbol{E}_1^{-1} \ldots \boldsymbol{E}_k^{-1} \Rightarrow \boldsymbol{A}^{-1} = \boldsymbol{E}_k \ldots \boldsymbol{E}_1$.

- The inverse of a product $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

$$(\boldsymbol{A}\boldsymbol{B})\boldsymbol{B}^{-1}\boldsymbol{A}^{-1} = \boldsymbol{A}(\boldsymbol{B}\boldsymbol{B}^{-1})\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{I}\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$$

$$\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}(\boldsymbol{A}\boldsymbol{B}) = \boldsymbol{B}^{-1}(\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{B} = \boldsymbol{B}^{-1}\boldsymbol{I}\boldsymbol{B} = \boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{I}$$

- If $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ invertible so are: $c\boldsymbol{A}$, $\boldsymbol{A}^T$, $\boldsymbol{A}^k$

$$(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$$

Verify

$$\boldsymbol{A}^T(\boldsymbol{A}^{-1})^T = (\boldsymbol{A}^{-1}\boldsymbol{A})^T = \boldsymbol{I}$$

$$(\boldsymbol{A}^{-1})^T\boldsymbol{A}^T = (\boldsymbol{A}\boldsymbol{A}^{-1})^T = \boldsymbol{I}$$