- New concepts:
  - Vector space
  - Vector subspace
  - Span of a set of vectors
  - Linear dependence and independence

- Formalize linear combinations

Addition rules for	$\forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V$
$\boldsymbol{a}+\boldsymbol{b} \in V$	Closure
$\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})=(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}$	Associativity
$\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$	Commutativity
$\boldsymbol{0}+\boldsymbol{a}=\boldsymbol{a}$	Zero vector
$\boldsymbol{a}+(-\boldsymbol{a})=\boldsymbol{0}$	Additive inverse
Scaling rules for	$\forall \boldsymbol{a}, \boldsymbol{b} \in V$, $\forall x, y \in S$
$x\boldsymbol{a} \in V$	Closure
$x(\boldsymbol{a}+\boldsymbol{b})=x\boldsymbol{a}+x\boldsymbol{b}$	Distributivity
$(x+y)\boldsymbol{a}=x\boldsymbol{a}+y\boldsymbol{a}$	Distributivity
$x(y\boldsymbol{a})=(xy)\boldsymbol{a}$	Composition
$1\boldsymbol{a}=\boldsymbol{a}$	Scalar identity

Table 1. Vector space properties

- Example: $V=\mathbb{R}^m, S=\mathbb{R}$

- Consider projection in $\mathbb{R}^2$ onto the x_1 axis

$$\boldsymbol{P} = \boldsymbol{e}_1 \boldsymbol{e}_1^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{P}\boldsymbol{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

- Let $V = \mathbb{R}^2$ and consider U be the set of all vectors in $\mathbb{R}^2$ with zero second component. Notice that $U \subset V$ and U is also a vector space, i.e., any linear combination of vectors in U stays within U
- In general if vectors in V form a vector space, $U \subset V$ and if for any $\alpha, \beta \in \mathbb{R}$, $\boldsymbol{u}, \boldsymbol{v} \in U$, $\alpha \boldsymbol{u} + \beta \boldsymbol{v} \in U$, then U is a vector subspace of V
- Example

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x_1, x_2 \in \mathbb{R} \right\}, U = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, x_1 \in \mathbb{R} \right\}$$

  Note that vectors in U still have two components, just like those in V
- Choose $\alpha = -\beta$, $\boldsymbol{u} = \boldsymbol{v}$ to set that $\alpha \boldsymbol{u} + \beta \boldsymbol{v} = \alpha \boldsymbol{u} - \alpha \boldsymbol{u} = \boldsymbol{0}$ must be within the subspace, i.e., the zero element is always a member of a subspace

Definition. *The* span *of vectors* $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in \mathcal{V}$, *is the set of vectors reachable by linear combination*

$$\operatorname{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\} = \{\boldsymbol{b} \in \mathcal{V} \mid \exists x_1, \ldots, x_n \in \mathcal{S} \text{ such that } \boldsymbol{b} = x_1 \boldsymbol{a}_1 + \ldots x_n \boldsymbol{a}_n\}.$$

The notation used for set on the right hand side is read: "those vectors $\boldsymbol{b}$ in $\mathcal{V}$ with the property that there exist n scalars $x_1, \ldots, x_n$ to obtain $\boldsymbol{b}$ by linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$.

- A linear combination is conveniently expressed as a matrix-vector product leading to a different formulation of the same concept

Definition. *The* column space *(or* range*) of matrix* $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ *is the set of vectors reachable by linear combination of the matrix column vectors*

$$C(\boldsymbol{A}) = \operatorname{range}(\boldsymbol{A}) = \{\boldsymbol{b} \in \mathbb{R}^m \mid \exists \boldsymbol{x} \in \mathbb{R}^n \text{ such that } \boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}\} \subseteq \mathbb{R}^m$$

In the example (?)

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\operatorname{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\} = \operatorname{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$$

since $\boldsymbol{a}_3 = \boldsymbol{a}_1 + \boldsymbol{a}_2 \Leftrightarrow \boldsymbol{a}_1 + \boldsymbol{a}_2 - \boldsymbol{a}_3 = \boldsymbol{0}$. Introduce a concept to capture the idea that a vector can be expressed in terms of other vectors.

Definition. *The vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in \mathcal{V}$, are linearly dependent if there exist n scalars, $x_1, \ldots, x_n \in \mathcal{S}$, at least one of which is different from zero such that*

$$x_1\boldsymbol{a}_1 + \ldots x_n\boldsymbol{a}_n = \boldsymbol{0}$$

Note that $\{\boldsymbol{0}\}$, with $\boldsymbol{0} \in \mathcal{V}$ is a linearly dependent set of vectors since $1 \cdot \boldsymbol{0} = \boldsymbol{0}$.

The converse of linear dependence is linear independence, a member of the set cannot be expressed as a non-trivial linear combination of the other vectors

Definition. *The vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in \mathcal{V}$, are linearly independent if the only n scalars, $x_1, \ldots, x_n \in \mathcal{S}$, that satisfy*

$$x_1 \boldsymbol{a}_1 + \ldots x_n \boldsymbol{a}_n = \boldsymbol{0}, \tag{1}$$

are $x_1 = 0$, $x_2 = 0$,...,$x_n = 0$.

The choice $\boldsymbol{x} = \begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix}^T = \boldsymbol{0}$ that always satisfies (1) is called a *trivial solution*. We can restate linear independence as (1) being satisfied *only* by the trivial solution.