- New concepts:
  - Span of a set of vectors, matrix column space
  - Matrix null space
  - Matrix row space
  - Matrix left null space
  - Vector space basis
  - Vector space sums

Definition. *The span of vectors* $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in \mathcal{V}$, *is the set of vectors reachable by linear combination*

$$\operatorname{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\} = \{\boldsymbol{b} \in \mathcal{V} \,|\, \exists x_1, \ldots, x_n \in \mathcal{S} \text{ such that } \boldsymbol{b} = x_1 \boldsymbol{a}_1 + \ldots x_n \boldsymbol{a}_n\}.$$

The notation used for set on the right hand side is read: "those vectors $\boldsymbol{b}$ in $\mathcal{V}$ with the property that there exist n scalars $x_1, \ldots, x_n$ to obtain $\boldsymbol{b}$ by linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$.

- A linear combination is conveniently expressed as a matrix-vector product leading to a different formulation of the same concept

Definition. *The column space (or range) of matrix* $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ *is the set of vectors reachable by linear combination of the matrix column vectors*

$$C(\boldsymbol{A}) = \operatorname{range}(\boldsymbol{A}) = \{\boldsymbol{b} \in \mathbb{R}^m \,|\, \exists \boldsymbol{x} \in \mathbb{R}^n \text{ such that } \boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}\} \subseteq \mathbb{R}^m$$

Introduce a characterization of the column vectors of a matrix related to linear dependence

Definition. *The null space of a matrix* $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ *is the set*

$$N(\boldsymbol{A}) = \mathrm{null}(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\} \subseteq \mathbb{R}^n$$

- If $\mathrm{null}(\boldsymbol{A}) = \{\boldsymbol{0}\}$ then the column vectors of $\boldsymbol{A}$ are linearly independent, since the only way to satisfy (?) is by the trivial solution $\boldsymbol{x} = \boldsymbol{0}$

For example $\boldsymbol{A} = [\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \boldsymbol{a}_3\ ]$ below, $c\,(\boldsymbol{a}_1 + \boldsymbol{a}_2 - \boldsymbol{a}_3) = \boldsymbol{0}$ for any scalar c, hence

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \Rightarrow C(\boldsymbol{A}) = \mathrm{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2\}, N(\boldsymbol{A}) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$$

.

Recall definitions of column space, null space of $\boldsymbol{A} \in \mathbb{R}^{m \times n}$

$$
\begin{aligned}
C(\boldsymbol{A}) &= \{\boldsymbol{b} \in \mathbb{R}^m | \exists \boldsymbol{x} \in \mathbb{R}^n \text{ such that } \boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}\} \subseteq \mathbb{R}^m \\
N(\boldsymbol{A}) &= \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\} \subseteq \mathbb{R}^n
\end{aligned}
$$

Note that $C(\boldsymbol{A}) \subseteq \mathbb{R}^m$, $N(\boldsymbol{A}) \subseteq \mathbb{R}^n$ means that $C(\boldsymbol{A})$, $N(\boldsymbol{A})$ are subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ respectively. In fact, we can make a stronger statement, that they are vector subspaces

$$
C(\boldsymbol{A}) \leq \mathbb{R}^m, N(\boldsymbol{A}) \leq \mathbb{R}^n
$$

Proof. Let $\boldsymbol{u}, \boldsymbol{v} \in C(\boldsymbol{A})$, $\alpha, \beta \in \mathcal{S}$. By defintion of $C(\boldsymbol{A})$ there exist $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ such that $\boldsymbol{u} = \boldsymbol{A}\boldsymbol{x}$ and $\boldsymbol{v} = \boldsymbol{A}\boldsymbol{y}$. Using vector space properties

$$
\alpha\boldsymbol{u} + \beta\boldsymbol{v} = \alpha\boldsymbol{A}\boldsymbol{x} + \beta\boldsymbol{A}\boldsymbol{y} = \boldsymbol{A}(\alpha\boldsymbol{x} + \beta\boldsymbol{y}),
$$

hence $\alpha\boldsymbol{u} + \beta\boldsymbol{v} \in C(\boldsymbol{A})$ (it is obtained as the image through the linear mapping $\boldsymbol{A}$ of $\alpha\boldsymbol{x} + \beta\boldsymbol{y}$)

- Recall that if $\boldsymbol{u}^T\boldsymbol{v}=0$, with $\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^m$ then $\boldsymbol{u}\perp\boldsymbol{v}$ (orthogonal)

Proposition. *If $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_n\in\mathbb{R}^m$ are non-zero ($\boldsymbol{u}_i\neq\boldsymbol{0}$) and pairwise orthogonal, $\boldsymbol{u}_i^T\boldsymbol{u}_j=0$ for $i\neq j$ then they form a linearly independent set of vectors.*

Proof. *Consider the equation equating the linear combination $c_1\boldsymbol{u}_1+\ldots+c_n\boldsymbol{u}_n$ to the zero vector*

$$c_1\boldsymbol{u}_1+\ldots+c_n\boldsymbol{u}_n=\boldsymbol{0} \tag{1}$$

Multiply on the left by $\boldsymbol{u}_i^T$ and use orthogonality to obtain $c_i=0$ for $i=1,\ldots,n$. The only solution to (1) is $c_1=c_2=\ldots=c_n=0$, hence $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_n\}$ is a linearly independent set.

- For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, seen as a linear mapping $\boldsymbol{A}: \mathbb{R}^n \mapsto \mathbb{R}^m$, that given input vector $\boldsymbol{x} \in \mathbb{R}^n$ returns output vector $\boldsymbol{b} \in \mathbb{R}^m, \boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$, we have defined the vector space of possible outputs, the column space of $\boldsymbol{A}$

$$C(\boldsymbol{A}) = \{\boldsymbol{b} \in \mathbb{R}^m | \, \exists \boldsymbol{x} \in \mathbb{R}^n \, \text{such that} \, \boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}\} \subseteq \mathbb{R}^m$$

- The transpose $\boldsymbol{A}^T \in \mathbb{R}^{n \times m}$ can also be seen as a linear mapping. Given some input vector $\boldsymbol{y} \in \mathbb{R}^m$ the mapping returns the output vector $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{c} = \boldsymbol{A}^T \boldsymbol{y}$. The set of possible outputs is the column space of $\boldsymbol{A}^T$. Since columns of $\boldsymbol{A}^T$ are rows of $\boldsymbol{A}$, we can define the *row space* of $\boldsymbol{A}$ as

$$R(\boldsymbol{A}) = C(\boldsymbol{A}^T) = \{\boldsymbol{c} \in \mathbb{R}^n | \, \exists \boldsymbol{y} \in \mathbb{R}^m \, \text{such that} \, \boldsymbol{c} = \boldsymbol{A}^T \boldsymbol{y}\} \subseteq \mathbb{R}^n$$

- *Left null space*, $N(\boldsymbol{A}^T) = \{\boldsymbol{y} \in \mathbb{R}^m | \, \boldsymbol{A}^T \boldsymbol{y} = 0\} \subseteq \mathbb{R}^m$, the part of $\mathbb{R}^m$ *not reachable* by linear combination of columns of $\boldsymbol{A}$

Definition. *A set of vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in \mathcal{V}$ is a basis for vector space $\mathcal{V}$ if:*

1. $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ *are linearly independent;*
2. $\operatorname{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\} = \mathcal{V}$.

Definition. *The number of vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in \mathcal{V}$ within a basis is the dimension of the vector space $\mathcal{V}$.*

Definition. *Given two vector subspaces* $(\mathcal{U},\mathcal{S},+)$, $(\mathcal{V},\mathcal{S},+)$ *of the space* $(\mathcal{W},\mathcal{S},+)$, *the sum is the set* $\mathcal{U}+\mathcal{V}=\{\boldsymbol{u}+\boldsymbol{v}|\ \boldsymbol{u}\in\mathcal{U},\boldsymbol{v}\in\mathcal{V}\}$.

Definition. *Given two vector subspaces* $(\mathcal{U},\mathcal{S},+)$, $(\mathcal{V},\mathcal{S},+)$ *of the space* $(\mathcal{W},\mathcal{S},+)$, *the direct sum is the set* $\mathcal{U}\oplus\mathcal{V}=\{\boldsymbol{u}+\boldsymbol{v}|\ \exists!\boldsymbol{u}\in\mathcal{U},\exists!\boldsymbol{v}\in\mathcal{V}\}$. *(unique decomposition)*

Definition. *Given two vector subspaces* $(\mathcal{U},\mathcal{S},+)$, $(\mathcal{V},\mathcal{S},+)$ *of the space* $(\mathcal{W},\mathcal{S},+)$, *the intersection is the set*

$$\mathcal{U}\cap\mathcal{V}=\{\boldsymbol{x}|\boldsymbol{x}\in\mathcal{U},\boldsymbol{x}\in\mathcal{V}\}.$$

Definition. *Two vector subspaces* $(\mathcal{U},\mathcal{S},+)$, $(\mathcal{V},\mathcal{S},+)$ *of the space* $(\mathcal{W},\mathcal{S},+)$ *are orthogonal subspaces, denoted* $\mathcal{U}\perp\mathcal{V}$ *if* $\boldsymbol{u}^T\boldsymbol{v}=0$ *for any* $\boldsymbol{u}\in\mathcal{U},\boldsymbol{v}\in\mathcal{V}$.

Definition. *Two vector subspaces* $(\mathcal{U},\mathcal{S},+)$, $(\mathcal{V},\mathcal{S},+)$ *of the space* $(\mathcal{W},\mathcal{S},+)$ *are orthogonal complements, denoted* $\mathcal{U}=\mathcal{V}^\perp$, $\mathcal{V}=\mathcal{U}^\perp$ *if they are orthogonal subspaces and* $\mathcal{U}\cap\mathcal{V}=\{\mathbf{0}\}$, *i.e., the null vector is the only common element of both subspaces.*