- New concepts:
  - Orthonormal vector set
  - Transforming a basis set into an orthonormal set by Gram-Schmidt
  - $\boldsymbol{QR}$ factorization of a matrix
  - Orthonormal bases for column, null space

Definition. *The Dirac delta symbol δ_{ij} is defined as*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Definition. *A set of vectors $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ is said to be orthonormal if*

$$\boldsymbol{u}_i^T \boldsymbol{u}_j = \delta_{ij}$$

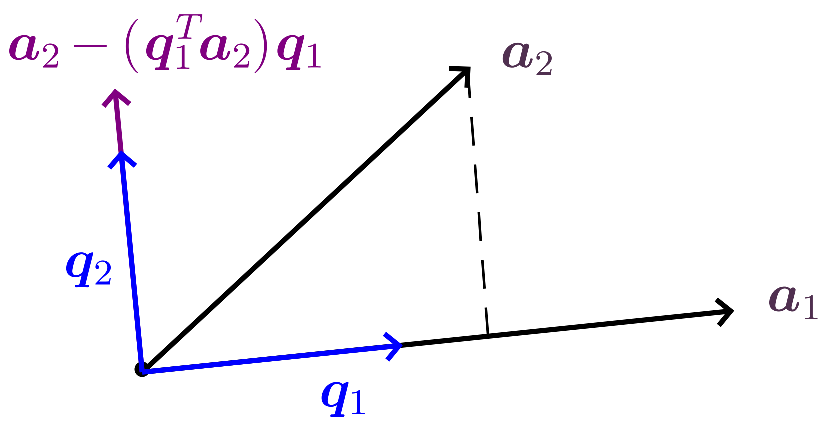
- The column vectors of the identity matrix are orthonormal

$$\boldsymbol{I} = \begin{pmatrix} \boldsymbol{e}_1 & \ldots & \boldsymbol{e}_m \end{pmatrix}, \boldsymbol{e}_i^T \boldsymbol{e}_j = \delta_{ij}, \boldsymbol{I}^T \boldsymbol{I} = \boldsymbol{I}$$

- Columns of $\boldsymbol{Q} = [\begin{array}{ccc} \boldsymbol{q}_1 & \ldots & \boldsymbol{q}_n \end{array}] \in \mathbb{R}^{m \times n}$ are orthonormal if

$$\boldsymbol{Q}^T \boldsymbol{Q} = \begin{bmatrix} \boldsymbol{q}_1^T \\ \vdots \\ \boldsymbol{q}_n^T \end{bmatrix} [\begin{array}{ccc} \boldsymbol{q}_1 & \ldots & \boldsymbol{q}_n \end{array}] = \begin{bmatrix} \boldsymbol{q}_1^T \boldsymbol{q}_1 & \boldsymbol{q}_1^T \boldsymbol{q}_2 & \ldots & \boldsymbol{q}_1^T \boldsymbol{q}_n \\ \boldsymbol{q}_2^T \boldsymbol{q}_1 & \boldsymbol{q}_2^T \boldsymbol{q}_2 & \ldots & \boldsymbol{q}_2^T \boldsymbol{q}_n \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{q}_n^T \boldsymbol{q}_1 & \boldsymbol{q}_n^T \boldsymbol{q}_2 & \ldots & \boldsymbol{q}_n^T \boldsymbol{q}_n \end{bmatrix} = \boldsymbol{I}_n$$

- Transform columns of $\boldsymbol{A} = [\,\boldsymbol{a}_1 \;\; \dots \;\; \boldsymbol{a}_n\,]$ into $\boldsymbol{Q} = [\,\boldsymbol{q}_1 \;\; \dots \;\; \boldsymbol{q}_n\,]$, $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}_n$
- Idea:
  - Start with an arbitrary direction $\boldsymbol{a}_1$
  - Divide by its norm to obtain a unit-norm vector $\boldsymbol{q}_1 = \boldsymbol{a}_1 / \|\boldsymbol{a}_1\|$
  - Choose another direction $\boldsymbol{a}_2$
  - Subtract off its component along previous direction(s) $\boldsymbol{a}_2 - (\boldsymbol{q}_1^T\boldsymbol{a}_2)\boldsymbol{q}_1$
  - Divide by norm $\boldsymbol{q}_2 = (\boldsymbol{a}_2 - (\boldsymbol{q}_1^T\boldsymbol{a}_2)\boldsymbol{q}_1) / \|\boldsymbol{a}_2 - (\boldsymbol{q}_1^T\boldsymbol{a}_2)\boldsymbol{q}_1\|$
  - Repeat the above



- Observe that formulas such as

$$\boldsymbol{q}_2 = (\boldsymbol{a}_2 - (\boldsymbol{q}_1^T \boldsymbol{a}_2)\boldsymbol{q}_1) / \|\boldsymbol{a}_2 - (\boldsymbol{q}_1^T \boldsymbol{a}_2)\boldsymbol{q}_1\|$$

state: "$\boldsymbol{q}_2$ is obtained as a linear combination of $\boldsymbol{a}_2$ and $\boldsymbol{q}_1$"

- Use column structure of $\boldsymbol{A}, \boldsymbol{Q}$

$$\boldsymbol{A} = [\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \dots \ \ \boldsymbol{a}_n\ ] = [\ \boldsymbol{q}_1 \ \ \boldsymbol{q}_2 \ \ \dots \ \ \boldsymbol{q}_n\ ] \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & & \ddots & \\ 0 & 0 & \dots & r_{nn} \end{bmatrix} = \boldsymbol{Q}\boldsymbol{R}$$

- Identify on both sides to obtain

$$\begin{aligned} \boldsymbol{a}_1 &= r_{11}\,\boldsymbol{q}_1 \\ \boldsymbol{a}_2 &= r_{12}\,\boldsymbol{q}_1 + r_{22}\,\boldsymbol{q}_2 \\ \boldsymbol{a}_3 &= r_{13}\,\boldsymbol{q}_1 + r_{23}\,\boldsymbol{q}_2 + r_{33}\,\boldsymbol{q}_3 \\ &\vdots \\ \boldsymbol{a}_n &= r_{1n}\,\boldsymbol{q}_1 + r_{2n}\,\boldsymbol{q}_2 + r_{3n}\,\boldsymbol{q}_3 + \dots + r_{nn}\boldsymbol{q}_n \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_1 &= \boldsymbol{a}_1 / r_{11} \\ \boldsymbol{q}_2 &= (\boldsymbol{a}_2 - r_{12}\,\boldsymbol{q}_1) / r_{22} \\ \boldsymbol{q}_3 &= (\boldsymbol{a}_3 - r_{13}\,\boldsymbol{q}_1 - r_{23}\,\boldsymbol{q}_2) / r_{33} \\ &\vdots \end{aligned}$$

Algorithm (Gram-Schmidt)

Given n vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$
Initialize $\boldsymbol{q}_1 = \boldsymbol{a}_1$,..,$\boldsymbol{q}_n = \boldsymbol{a}_n$, $\boldsymbol{R} = \boldsymbol{I}_n$
for $i = 1$ to n
  $r_{ii} = (\boldsymbol{q}_i^T \boldsymbol{q}_i)^{1/2}$; $\boldsymbol{q}_i = \boldsymbol{q}_i / r_{ii}$
  for $j = i$+1 to n
    $r_{ij} = \boldsymbol{q}_i^T \boldsymbol{a}_j$; $\boldsymbol{q}_j = \boldsymbol{q}_j - r_{ij} \boldsymbol{q}_i$
  end
end
return $\boldsymbol{Q}, \boldsymbol{R}$

- Matlab orth returns basis for $C(\boldsymbol{A})$

```
>> A=[1 -2 1 -1; 1 1 1 -1; 1 0 1 -1]; orth(A)
```

- Matlab null returns basis for $N(\boldsymbol{A})$

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$$\begin{pmatrix} 1 & 0.0 & 1 & -1 \\ 0.0 & 1 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}$$

```
>>
```