- New concepts:
  - Gaussian elimination as $\boldsymbol{LU}$
  - Gram-Schmidt as $\boldsymbol{QR}$
  - Solving linear systems

Recall from Lesson 9

- Gaussian elimination produces a sequence matrices similar to $A \in \mathbb{R}^{m\times m}$

$$A = A^{(0)} \sim A^{(1)} \sim \cdots \sim A^{(k)} \sim \cdots \sim A^{(m-1)}$$

- Step k produces zeros underneath diagonal position (k,k)
- Step k can be represented as multiplication by matrix

$$A^{(k)} = L_k A^{(k-1)}, L_k = \begin{pmatrix} 1 & \ldots & 0 & \ldots & 1 \\ 0 & \ddots & 0 & \ldots & 0 \\ 0 & \ldots & 1 & \ldots & 0 \\ 0 & \ldots & -l_{k+1,k} & \ldots & 0 \\ \vdots & \ldots & \vdots & \ddots & \vdots \\ 0 & \ldots & -l_{m,k} & \ldots & 1 \end{pmatrix}, l_{j,k} = \frac{a_{j,k}^{(k-1)}}{a_{k,k}^{(k-1)}}, A^{(k)} = [a_{i,j}^{(k)}]$$

- All $m-1$ steps produce an upper triangular matrix

$$L_{m-1} \ldots L_2 L_1 A = U \Rightarrow A = L_1^{-1} L_2^{-1} \ldots L_{m-1}^{-1} U = LU$$

- With permutations $PA = LU$ (Matlab `[L,U,P]=lu(A), A=P'*L*U`)

- With known LU-factorization: $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}\Rightarrow(\boldsymbol{L}\boldsymbol{U})\boldsymbol{x}=\boldsymbol{P}\boldsymbol{b}\Rightarrow\boldsymbol{L}(\boldsymbol{U}\boldsymbol{x})=\boldsymbol{P}\boldsymbol{b}$
- To solve $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$:
  1. Carry out LU-factorization: $\boldsymbol{P}^T\boldsymbol{L}\boldsymbol{U}=\boldsymbol{A}$
  2. Solve $\boldsymbol{L}\boldsymbol{y}=\boldsymbol{c}=\boldsymbol{P}\boldsymbol{b}$ by forward substitution to find $\boldsymbol{y}$
  3. Solve $\boldsymbol{U}\boldsymbol{x}=\boldsymbol{y}$ by backward substitution
- FLOP = floating point operation = one multiplication and one addition
- Operation counts: how many FLOPS in each step?
  1. Each $\boldsymbol{L}_k\boldsymbol{A}^{(k-1)}$ costs $(m-k)^2$ FLOPS. Overall

$$(m-1)^2+(m-2)^2+\cdots+1^2=\frac{m(m-1)(2m-1)}{6}\approx\frac{m^3}{3}$$

  2. Forward substitution step k costs k flops

$$1+2+\cdots+m=\frac{m(m+1)}{2}\approx\frac{m^2}{2}$$

  3. Backward substitution cost is identical $m(m+1)/2\approx m^2/2$/

Recall from Lesson 13

- Orthonormalization of columns of $\boldsymbol{A}$ is also a factorization

$$\boldsymbol{A} = [\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \dots \ \ \boldsymbol{a}_n\ ] = [\ \boldsymbol{q}_1 \ \ \boldsymbol{q}_2 \ \ \dots \ \ \boldsymbol{q}_n\ ] \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & & \ddots & \\ 0 & 0 & \dots & r_{nn} \end{bmatrix} = \boldsymbol{Q}\boldsymbol{R}$$

$$\begin{aligned} \boldsymbol{a}_1 &= r_{11}\,\boldsymbol{q}_1 \\ \boldsymbol{a}_2 &= r_{12}\,\boldsymbol{q}_1 + r_{22}\,\boldsymbol{q}_2 \\ \boldsymbol{a}_3 &= r_{13}\,\boldsymbol{q}_1 + r_{23}\,\boldsymbol{q}_2 + r_{33}\,\boldsymbol{q}_3 \\ &\vdots \\ \boldsymbol{a}_n &= r_{1n}\,\boldsymbol{q}_1 + r_{2n}\,\boldsymbol{q}_2 + r_{3n}\,\boldsymbol{q}_3 + \dots + r_{nn}\boldsymbol{q}_n \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_1 &= \boldsymbol{a}_1 / r_{11} \\ \boldsymbol{q}_2 &= (\boldsymbol{a}_2 - r_{12}\,\boldsymbol{q}_1) / r_{22} \\ \boldsymbol{q}_3 &= (\boldsymbol{a}_3 - r_{13}\,\boldsymbol{q}_1 - r_{23}\,\boldsymbol{q}_2) / r_{33} \\ &\vdots \end{aligned}$$

- Operation count:
  - $r_{jk} = \boldsymbol{q}_j^T \boldsymbol{a}_k$ costs m FLOPS
  - There are $1 + 2 + \dots + n$ components in $\boldsymbol{R}$, Overall cost $n(n+1)m/2$
- With permutations $\boldsymbol{A}\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{R}$ (Matlab `[Q,R,P]=qr(A)`)

- With known QR-factorization: $\boldsymbol{Ax}=\boldsymbol{b}\Rightarrow(\boldsymbol{QRP}^T)\boldsymbol{x}=\boldsymbol{b}\Rightarrow\boldsymbol{Ry}=\boldsymbol{Q}^T\boldsymbol{b}$
- To solve $\boldsymbol{Ax}=\boldsymbol{b}$:
  1. Carry out QR-factorization: $\boldsymbol{QRP}^T=\boldsymbol{A}$
  2. Compute $\boldsymbol{c}=\boldsymbol{Q}^T\boldsymbol{b}$
  3. Solve $\boldsymbol{Ry}=\boldsymbol{c}$ by backward substitution
  4. Find $\boldsymbol{x}=\boldsymbol{P}^T\boldsymbol{y}$
- Operation counts: how many FLOPS in each step?
  1. QR-factorization $m^2(m+1)/2\approx m^3/2$
  2. Compute $\boldsymbol{c}$, m^2
  3. Backward substitution $m(m+1)/2\approx m^2/2$