- New concepts:
  - Geometric definition: volume of a hyper-parallelipiped
  - Determinant calculation rules
  - Algebraic definition: sum over permutations
  - Example of a bad algorithm: Cramer's rule

Definition. *The determinant of a square matrix* $\boldsymbol{A} = [\ \boldsymbol{a}_1 \ \dots \ \boldsymbol{a}_m\ ] \in \mathbb{R}^{m\times m}$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{vmatrix} \in \mathbb{R}$$

is a real number giving the (oriented) volume of the parallelepiped spanned by matrix column vectors.

- $m = 2$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \det(\boldsymbol{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

- $m = 3$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \det(\boldsymbol{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

- Computation of a determinant with $m=2$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

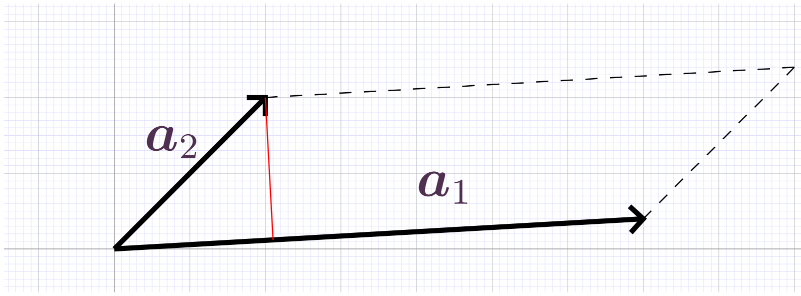
- Computation of a determinant with $m=3$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}$$

$$-a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

- Where do these determinant computation rules come from? Three viewpoints
  - *Geometry viewpoint*: determinants express parallelepiped volumes
  - *Algebra viewpoint*: determinants are computed from all possible products that can be formed from choosing a factor from each row and each column
  - *Function viewpoint*: the determinant is a function satisfying certain properties (see Textbook Definition 3.2.1).

- $m = 2$



$$\boldsymbol{A} = [\,\boldsymbol{a}_1 \;\; \boldsymbol{a}_2\,] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- In two dimensions a "parallelepiped" becomes a parallelogram with area given as

$$(\text{Area}) = (\text{Length of Base}) \times (\text{Length of Height})$$

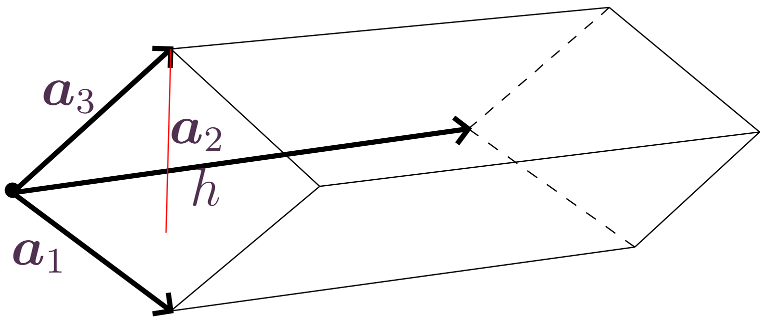
- Take $\boldsymbol{a}_1$ as the base, with length $b = \|\boldsymbol{a}_1\|$. Vector $\boldsymbol{a}_1$ is at angle φ_1 to x_1-axis, $\boldsymbol{a}_2$ is at angle φ_2 to x_2-axis, and the angle between $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ is $\theta = \varphi_2 - \varphi_1$. The height has length

$$h = \|\boldsymbol{a}_2\| \sin\theta = \|\boldsymbol{a}_2\| \sin(\varphi_2 - \varphi_1) = \|\boldsymbol{a}_2\| (\sin\varphi_2 \cos\varphi_1 - \sin\varphi_1 \cos\varphi_2)$$

- Use $\cos\varphi_1 = a_{11} / \|\boldsymbol{a}_1\|$, $\sin\varphi_1 = a_{12} / \|\boldsymbol{a}_1\|$, $\cos\varphi_2 = a_{21} / \|\boldsymbol{a}_2\|$, $\sin\varphi_2 = a_{22} / \|\boldsymbol{a}_2\|$

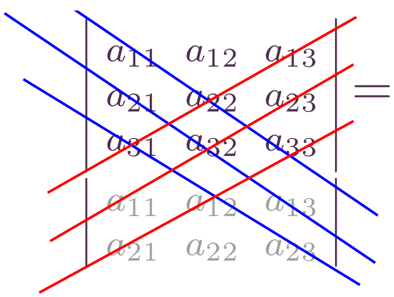
$$(\text{Area}) = \|\boldsymbol{a}_1\| \, \|\boldsymbol{a}_2\| (\sin\varphi_2 \cos\varphi_1 - \sin\varphi_1 \cos\varphi_2) = a_{11} a_{22} - a_{12} a_{21}$$

- $m=3$, $\boldsymbol{A}=[\,\boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \boldsymbol{a}_3\,]$



The volume is (area of base) $\times$ (height) and given as the value of the determinant

$$\det(\boldsymbol{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$



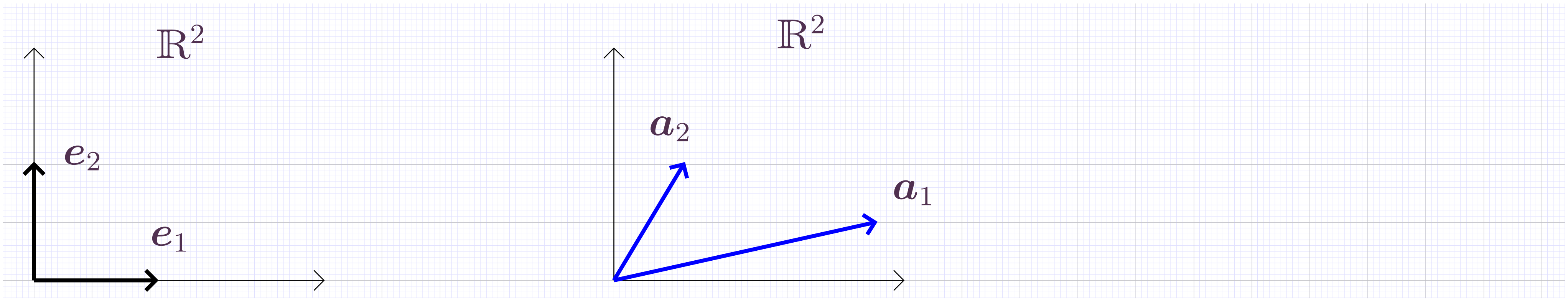
$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$

Sarrus rule (3-determinant mnemonic)

- Recall that $\boldsymbol{A} = [\ \boldsymbol{a}_1 \ \ \ldots \ \ \boldsymbol{a}_m\ ] \in \mathbb{R}^{m\times m}$ defines a linear mapping $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$
- Observe

$$T(\boldsymbol{e}_1) = \boldsymbol{A}\boldsymbol{e}_1 = \boldsymbol{a}_1, \ldots, T(\boldsymbol{e}_m) = \boldsymbol{A}\boldsymbol{e}_m = \boldsymbol{a}_m$$

- Since $T: \mathbb{R}^m \to \mathbb{R}^m$, interpret above to mean that $\det(\boldsymbol{A})$ measures the ratio of the volume of the hyperparallelipiped $[\ \boldsymbol{a}_1 \ \ \ldots \ \ \boldsymbol{a}_m\ ]$ w.r.t. unit hypercube



- From above, observe that the determinant can be used to “measure” both a matrix and a linear mapping.

- The geometric interpretation of a determinant as an oriented volume is useful in establishing rules for calculation with determinants:
  - Determinant of matrix with repeated columns is zero (since two edges of the parallelepiped are identical). Example for $m=3$

$$\Delta = \begin{vmatrix} a & a & u \\ b & b & v \\ c & c & w \end{vmatrix} = abw + bcu + cav - ubc - vca - wab = 0$$

    This is more easily seen using the column notation

$$\Delta = \det[\, \boldsymbol{a}_1 \;\; \boldsymbol{a}_1 \;\; \boldsymbol{a}_3 \;\; \ldots \,] = 0$$

  - Determinant of matrix with linearly dependent columns is zero (since one edge lies in the 'hyperplane' formed by all the others)

- Separating sums in a column (similar for rows). With $\boldsymbol{a}_i, \boldsymbol{b}_1 \in \mathbb{R}^m$:

$$\det[\ \boldsymbol{a}_1 + \boldsymbol{b}_1 \ \ \boldsymbol{a}_2 \ \ \ldots \ \ \boldsymbol{a}_m\ ] = \det[\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \ldots \ \ \boldsymbol{a}_m\ ] + \det[\ \boldsymbol{b}_1 \ \ \boldsymbol{a}_2 \ \ \ldots \ \ \boldsymbol{a}_m\ ]$$

- Scalar product in a column (similar for rows). With $\alpha \in \mathbb{R}$:

$$\det[\ \alpha \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \ldots \ \ \boldsymbol{a}_m\ ] = \alpha \det \ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \ldots \ \ \boldsymbol{a}_m$$

- Linear combinations of columns (similar for rows). With $\alpha \in \mathbb{R}$:

$$\det[\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \ldots \ \ \boldsymbol{a}_m\ ] = \det[\ \boldsymbol{a}_1 \ \ \alpha \boldsymbol{a}_1 + \boldsymbol{a}_2 \ \ \ldots \ \ \boldsymbol{a}_m\ ]$$

- Swapping two columns (or rows) changes determinant sign

$$\det[\ \boldsymbol{a}_1 \ \ \ldots \ \ \boldsymbol{a}_i \ \ \ldots \ \ \boldsymbol{a}_j \ \ \ldots \ \ \boldsymbol{a}_m\ ] = -\det[\ \boldsymbol{a}_1 \ \ \ldots \ \ \boldsymbol{a}_j \ \ \ldots \ \ \boldsymbol{a}_i \ \ \ldots \ \ \boldsymbol{a}_m\ ]$$

- For $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, the (i,j)-*minor* of $\boldsymbol{A}$, $m_{i,j}$, is the $(m-1) \times (m-1)$ determinant obtained by deleting row i and column j

$$m_{i,j} = \begin{vmatrix} a_{11} & \dots & a_{1,j-1} & a_{1,j+1} & \dots & a_{1,m} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j+1} & \dots & a_{i-1,m} \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j+1} & \dots & a_{i+1,m} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ a_{m,1} & \dots & a_{m,j-1} & a_{m,j+1} & \dots & a_{m,m} \end{vmatrix}$$

- The (i,j)-*cofactor* of $\boldsymbol{A}$ is

$$c_{i,j} = (-1)^{i+j} m_{i,j}$$

- Minors and cofactors are useful in determinant calculations

- A determinant of size m can be expressed as a sum of determinants of size $m-1$ by expansion along a row or column

$$
\begin{vmatrix}
a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\
a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \dots & a_{mm}
\end{vmatrix}
= a_{11}\begin{vmatrix}
a_{22} & a_{23} & \dots & a_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m2} & a_{m3} & \dots & a_{mm}
\end{vmatrix} -
a_{12}\begin{vmatrix}
a_{21} & a_{23} & \dots & a_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m3} & \dots & a_{mm}
\end{vmatrix} +
\dots
+(-1)^{m+1}a_{1m}\begin{vmatrix}
a_{21} & a_{23} & \dots & a_{2,m-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m3} & \dots & a_{m,m-1}
\end{vmatrix}
$$

- Expressed as cofactors $\det(\boldsymbol{A}) = \sum_{j=1}^{m} a_{1j}\, c_{1j}$
- In general $\det(\boldsymbol{A}) = \sum_{j=1}^{m} a_{i,j}\, c_{i,j} = \sum_{i=1}^{m} a_{i,j}\, c_{i,j}$

- Determinant rule

$$\det(\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ \ \ldots\ \ \boldsymbol{a}_m\ ) = \det(\ \boldsymbol{a}_1\ \ \alpha\boldsymbol{a}_1 + \boldsymbol{a}_2\ \ \ldots\ \ \boldsymbol{a}_m\ )$$

  means that Gaussian elimination operations can be used to compute $\det(\boldsymbol{A})$
- Approach:
  - carry out linear combination of rows to reduce $\boldsymbol{A}$ to triangular form

$$\det(\boldsymbol{A}) = \det(\boldsymbol{U})$$

  - The determinant of a triangular matrix is simply the product of its diagonal elements due to cofactor expansion

$$\det(\boldsymbol{U}) = \begin{vmatrix} u_{11} & u_{12} & \ldots & u_{1,m} \\ 0 & u_{22} & \ldots & u_{2,m} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \ldots & u_{m,m} \end{vmatrix} = u_{11} \begin{vmatrix} u_{22} & \ldots & u_{2,m} \\ \vdots & \ddots & \\ 0 & \ldots & u_{m,m} \end{vmatrix} = u_{11}u_{22}\ldots u_{\text{mm}}$$

- The above leads to $\mathcal{O}(m^3/3)$ FLOPs computations for a determinant

- $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$

  Proof: After presentation of SVD

- $\det(\boldsymbol{A}\boldsymbol{B}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$

  Proof: After presentation of SVD

- $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, if $\operatorname{rank}(\boldsymbol{A}) < m$ then $\det(\boldsymbol{A}) = 0$

  Motivation: Some of the column vectors are linearly dependent

- The formal definition of a determinant

$$\det A = \sum_{\sigma \in \Sigma} \nu(\sigma) a_{1i_1} a_{2i_2} \ldots a_{mi_m}$$

requires $mm!$ operations, a number that rapidly increases with m
- A more economical determinant is to use row and column combinations to create zeros and then reduce the size of the determinant, an algorithm reminiscent of Gauss elimination for systems

Example:

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ -2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 3 & 10 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & 10 \end{vmatrix} = 20 - 12 = 8$$

The first equality comes from linear combinations of rows, i.e. row 1 is added to row 2, and row 1 multiplied by 2 is added to row 3. These linear combinations maintain the value of the determinant. The second equality comes from expansion along the first column

- We've seen examples of "good" algorithms to solve a linear system (LU, QR)
- The above find a solution using $\mathcal{O}(m^3/2)$, $\mathcal{O}(m^3/2)$ FLOPS
- Consider now an example of a "bad" algorithm, known as Cramer's rule

$$x_i = \frac{\Delta_i}{\Delta}$$

- $\Delta = \det(\boldsymbol{A})$ is the *principal determinant* of the linear system
- $\Delta_i = \det([\ \boldsymbol{a}_1\ \ \dots\ \ \boldsymbol{a}_{i-1}\ \ \boldsymbol{b}\ \ \boldsymbol{a}_{i+1}\ \ \dots\ \ \boldsymbol{a}_m\ ])$ arises from minors of $\boldsymbol{A}$
- Why bad? It requires computation of $m+1$ determinants, each of which costs $\mathcal{O}(m^3/3)$ leading to $\mathcal{O}(m^4/3)$ overall
- Could be even worse! Before ubiquity of computers, determinants were often computed by the algebraic definition

$$\det A = \sum_{\sigma \in \Sigma} \nu(\sigma) a_{1i_1} a_{2i_2} \dots a_{mi_m}$$

  There are $m!$ terms in the sum. Each term costs m flops. Using this in Cramer's rule gives $\mathcal{O}(m^2\, m!)$ an incredibly large number for even small m.