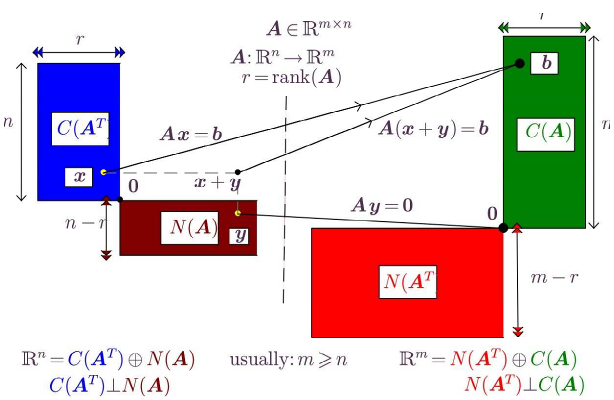
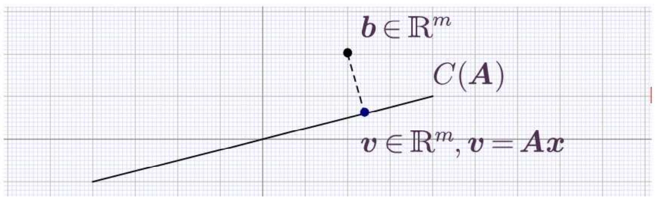
- New concepts:
  - The eigenvalue problem as one of the principal problems of linear algebra
  - Eigenvalues and eigenvectors
  - Characteristic polynomial
  - Simple cases
    - → diagonal matrices
    - → scaling matrices
    - → reflection matrices
    - → rotation matrices (complex eigenvalues, eigenvectors)

- Coordinates in a new basis, *solving a linear system* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{I}\boldsymbol{b}$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$
  1. Compute $\boldsymbol{L}\boldsymbol{U}$ factorization, $\boldsymbol{L}\boldsymbol{U} = \boldsymbol{A}$
  2. Solve $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$ by forward substitution
  3. Solve $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$ by backward substition
- Approximate $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{b} \cong \boldsymbol{A}\boldsymbol{x}$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, the *least squares problem*
  - Find $\boldsymbol{Q}\boldsymbol{R}$ factorization, $\boldsymbol{Q}\boldsymbol{R} = \boldsymbol{A}$. Projector onto $C(\boldsymbol{A}) = C(\boldsymbol{Q})$ is $\boldsymbol{P}_{\boldsymbol{Q}} = \boldsymbol{Q}\boldsymbol{Q}^T$
  - Projection of $\boldsymbol{b}$ onto $C(\boldsymbol{A})$ is $\boldsymbol{Q}\boldsymbol{Q}^T\boldsymbol{b}$. Set this equal to a linear combination of columns of $\boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{Q}^T\boldsymbol{b}$. Since $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$, solve the triangular system $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{Q}^T\boldsymbol{b}$ to find $\boldsymbol{x}$.





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- For square matrix $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ find *non-zero* vectors whose *directions* are not changed by multiplication by $\boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, λ is scalar, the *eigenvalue problem*.
- Consider the eigenproblem $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ for $\boldsymbol{A} \in \mathbb{R}^{m \times m}$. Rewrite as

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow (\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}.$$

  Since $\boldsymbol{x} \neq \boldsymbol{0}$, a solution to eigenproblem exists only if $\boldsymbol{A} - \lambda\boldsymbol{I}$ is singular.
- $\boldsymbol{A} - \lambda\boldsymbol{I}$ singular implies $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$.
- Investigate form of $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} - \lambda & a_{23} & \dots & a_{2m} \\ a_{31} & a_{32} & a_{33} - \lambda & \dots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mm} - \lambda \end{vmatrix}$$

- $p_m(\lambda) = \det(\lambda\boldsymbol{I} - \boldsymbol{A})$, an m^{th}-degree polynomial in λ, *characteristic polynomial* of $\boldsymbol{A}$, with m roots, $\lambda_1, \lambda_2, \dots, \lambda_m$, the eigenvalues of $\boldsymbol{A}$

- $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, eigenvalue problem $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ ($\boldsymbol{x} \neq \boldsymbol{0}$) in matrix form:

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{\Lambda}$$

$$\boldsymbol{X} = [\ \boldsymbol{x}_1 \quad \dots \quad \boldsymbol{x}_m\ ], \boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \dots, \lambda_m) = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_m \end{bmatrix}.$$

- $\boldsymbol{X}$ is the *eigenvector matrix*, $\boldsymbol{\Lambda}$ is the (diagonal) *eigenvalue matrix*
- If column vectors of $\boldsymbol{X}$ are linearly independent, then $\boldsymbol{X}$ is invertible

$$\boldsymbol{A} = \boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1},$$

  the *eigendecomposition* of $\boldsymbol{A}$ (compare to $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$, $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$)
- Rule "determinant of product = product of determinants" implies

$$\det(\boldsymbol{A}\boldsymbol{X}) = \det(\boldsymbol{X}\boldsymbol{\Lambda}) \Rightarrow \det(\boldsymbol{A}) = \det(\boldsymbol{\Lambda}) \text{ (for } \det(X) \neq 0).$$

- Eigendecomposition of $\boldsymbol{I} \in \mathbb{R}^{m \times m}$. Compare $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{\Lambda}$

$$\boldsymbol{I}\boldsymbol{I} = \boldsymbol{I}\boldsymbol{I}, \boldsymbol{A} = \boldsymbol{I}, \boldsymbol{X} = \boldsymbol{I}, \boldsymbol{\Lambda} = \boldsymbol{I}$$

  to find eigenvalues $\lambda_1 = 1, \ldots, \lambda_m = 1$, eigenvectors $\boldsymbol{x}_1 = \boldsymbol{e}_1, \ldots, \boldsymbol{x}_m = \boldsymbol{e}_m$.
- Eigendecomposition of $\boldsymbol{A} = \operatorname{diag}(s_1, s_2, \ldots, s_m)$. Compare $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{\Lambda}$

$$\boldsymbol{A}\boldsymbol{I} = \boldsymbol{I}\boldsymbol{A}$$

  to find eigenvalues $\lambda_1 = s_1, \ldots, \lambda_m = s_m$, eigenvectors $\boldsymbol{x}_1 = \boldsymbol{e}_1, \ldots, \boldsymbol{x}_m = \boldsymbol{e}_m$.
- Reflection across x_1-axis in $\mathbb{R}^2$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

  is a diagonal matrix, $\lambda_1 = 1$, $\lambda_2 = -1$, $\boldsymbol{x}_1 = \boldsymbol{e}_1$, $\boldsymbol{x}_2 = \boldsymbol{e}_2$

- Rotate by θ around x_3 axis in $\mathbb{R}^3$

$$\boldsymbol{A} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}, m = 3$$

- One direction not change by rotation is $\boldsymbol{x}_3 = \boldsymbol{e}_3$ with $\lambda_3 = 1$
- Where are the other two directions?
  - Compute characteristic polynomial $p_3(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A})$

$$p_3(\lambda) = \begin{vmatrix} \lambda - \cos\theta & \sin\theta & 0 \\ -\sin\theta & \lambda - \cos\theta & 0 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)(\lambda^2 - 2\lambda\cos\theta + 1)$$

  - One root of $p_3(\lambda)$ is $\lambda_3 = 1$, as expected.
  - Solve $\lambda^2 - 2\lambda\cos\theta + 1 = 0$ to find remaining eigenvalues to be *complex*

$$\lambda_{1,2} = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm i\sin\theta = e^{\pm i\theta} \in \mathbb{C}, i^2 = -1.$$

- $z \in \mathbb{C}$ can be represented in
  - Cartesian form $z = x + iy$
  - Polar form $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$
- Complex conjugate of $z \in \mathbb{C}$ negates imaginary part $\bar{z} = x - iy = re^{-i\theta}$
- Absolute value of $z \in \mathbb{C}$ is $|z| = (x^2 + y^2)^{1/2} = r$
- Argument of z is angle θ from polar form $z = re^{i\theta}$
- Absolute value can be expressed as $|z| = (\bar{z}\,z)^{1/2}$
- Recall for $\boldsymbol{x} \in \mathbb{R}^m$

$$\|\boldsymbol{x}\|_2^2 = \boldsymbol{x}^T\boldsymbol{x} = x_1^2 + \cdots + x_m^2,$$

  stating that squared 2-norm of real vector $\boldsymbol{x}$ is sum of squares of components.
- Extend above to vector of complex numbers $\boldsymbol{u} \in \mathbb{C}^m$ by

$$\|\boldsymbol{u}\|_2^2 = |u_1|^2 + \cdots + |u_m|^2 = (\bar{\boldsymbol{u}})^T\boldsymbol{u}.$$

- Taking the complex conjugate and transposing arises frequently, notation

$$\boldsymbol{u}^* = (\bar{\boldsymbol{u}})^T, \text{adjoint of } \boldsymbol{u}$$

- Consider $\lambda_2 = e^{i\theta} = \exp(i\theta) = \cos\theta + i\sin\theta$. Eigenvector $\boldsymbol{x}_2$ satisfies

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{x}_2 = \boldsymbol{0},$$

  which implies $\boldsymbol{x}_2 \in N(\boldsymbol{A} - \lambda_2 \boldsymbol{I})$
- Compute basis vector for $N(\boldsymbol{A} - \lambda_2 \boldsymbol{I})$

$$\boldsymbol{A} - \lambda_2 \boldsymbol{I} = \begin{bmatrix} -i\sin\theta & -\sin\theta & 0 \\ \sin\theta & -i\sin\theta & 0 \\ 0 & 0 & -e^{i\theta} \end{bmatrix} \sim \begin{bmatrix} -i\sin\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -e^{i\theta} \end{bmatrix}.$$

- Find eigenvector

$$\boldsymbol{x}_2 = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$$

- Repeat for $\lambda_2 = e^{-i\theta}$, find $\boldsymbol{x}_3 = \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$

- Compute $\boldsymbol{A}\boldsymbol{x}_2 - \lambda_2 \boldsymbol{x}_2 = (\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{x}_2$

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{x}_2 = \begin{bmatrix} -i\sin\theta & -\sin\theta & 0 \\ \sin\theta & -i\sin\theta & 0 \\ 0 & 0 & -e^{i\theta} \end{bmatrix} \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{0}. \checkmark$$

- In general a polynomial of degree m $p_m(\lambda)$ with real coefficients has m complext roots $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$

- Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- Eigenvalues $\lambda_1 = \lambda_2 = 1$, *a repeated root*, since

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda - 1 & 1 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2.$$

- However

$$\boldsymbol{A} - \lambda_1 \boldsymbol{I} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\operatorname{rank}(\boldsymbol{A} - \lambda_1 \boldsymbol{I}) = 1 = \dim C((\boldsymbol{A} - \lambda_1 \boldsymbol{I})^T)$, FTLA $\Rightarrow \dim N(\boldsymbol{A} - \lambda_1 \boldsymbol{I}) = 1$, only one non-zero eigenvector