- New concepts:
  - Algebraic, geometric multiplicities
  - Diagonalizability
  - Computing eigenvalues
  - Computing eigenvectors
  - Ill-conditioning of finding roots of characteristic polynomial
  - Diagonalizable matrices
  - Utility of diagonal representation

Definition 1. *The algebraic multiplicity of an eigenvalue* λ *is the number of times it appears as a repeated root of the characteristic polynomial* $p(\lambda) = \det(A - \lambda I)$

Example. $p(\lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$ has two single roots $\lambda_1 = 0$, $\lambda_2 = 1$ and a repeated root $\lambda_{3,4} = 2$. The eigenvalue $\lambda = 2$ has an algebraic multiplicity of 2

Definition 2. *The geometric multiplicity of an eigenvalue* λ *is the dimension of the null space of* $A - \lambda I$

Definition 3. *An eigenvalue for which the geometric multiplicity is less than the algebraic multiplicity is said to be defective*

Theorem. *A matrix is diagonalizable if the geometric multiplicity of each eigenvalue is equal to the algebraic multiplicity of that eigenvalue.*

- Finding eigenvalues as roots of characteristic polynomial $p(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A})$ is suitable for small matrices $\boldsymbol{A} \in \mathbb{R}^{m \times m}$.
  - analytical root-finding formulas are available only for $m \leqslant 4$
  - small errors in characteristic polynomial coefficients can lead to large errors in roots
- Octave/Matlab procedures to find characteristic polynomial
  - `poly(A)` function returns the coefficients
  - `roots(p)` function computes roots of the polynomial

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matlab>>A=[5 -4 2; 5 -4 1; -2 2 -3]; p=poly(A); disp(p)
matlab>>roots(p)'
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>>$(1 \;\; -2 \;\; -1)$

```
matlab>>
```

- Find eigenvectors as non-trivial solutions of system $(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{x}=0$, e.g., $\lambda_1=1$

$$\boldsymbol{A}-\lambda_1\boldsymbol{I}=\begin{pmatrix} 4 & -4 & 2 \\ 5 & -5 & 1 \\ -2 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} -2 & 2 & -4 \\ 0 & 0 & -6 \\ 5 & -5 & 1 \end{pmatrix} \sim \begin{pmatrix} -2 & 2 & -4 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

  Note convenient choice of row operations to reduce amount of arithmetic, and use of knowledge that $\boldsymbol{A}-\lambda_1\boldsymbol{I}$ is singular to deduce that last row must be null

- In traditional form the above row-echelon reduced system corresponds to

$$\left\{\begin{array}{rcl} -2x_1+2x_2-4x_3 & = & 0 \\ 0x_1+0x_2-6x_3 & = & 0 \\ 0x_1+0x_2+0x_3 & = & 0 \end{array}\right. \Rightarrow \boldsymbol{x}=\alpha\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \|\boldsymbol{x}\|=1 \Rightarrow \alpha=1/\sqrt{2}$$

- In Octave/Matlab the computations are carried out by the null function

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matlab>>null(A-eye(3))'
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>>$(\ -0.70711 \quad -0.70711 \quad 5.5511e-17\ )$

- *Ill-conditioning*: small errors in input produce large errors in output
- The eigenvalues of $\boldsymbol{I} \in \mathbb{R}^{3\times 3}$ are $\lambda_{1,2,3} = 1$, but small errors in numerical computation can give roots of the characteristic polynomial with imaginary parts

```
matlab>>roots(poly(eye(3)))'
```

- Avoid ill-conditioning of root finding by numerical methods (MATH566, MATH661)

```
matlab>>eig(eye(3))'
```

- Eigenvalue numerical methods use following properties:
  - $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow \boldsymbol{A}^{-1}\boldsymbol{x} = \lambda^{-1}\boldsymbol{x}$ if $\boldsymbol{A}^{-1}$ exists. "Inverse matrix has inverse eigenvalues"
  - $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow (\boldsymbol{A} + \mu\boldsymbol{I})\boldsymbol{x} = (\lambda + \mu)\boldsymbol{x}$. "Shifted matrix has shifted eigenvalues"
  - $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, $\boldsymbol{x} = \boldsymbol{B}\boldsymbol{y} \Rightarrow \boldsymbol{A}\boldsymbol{B}\boldsymbol{y} = \lambda\boldsymbol{B}\boldsymbol{y} \Rightarrow \boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B}\boldsymbol{y} = \lambda\boldsymbol{y}$, if $\boldsymbol{B}^{-1}$ exists
- Matrix $\boldsymbol{A}$ is *similar* to matrix $\boldsymbol{C}$ if there exists $\boldsymbol{B}$ nonsingular for which $\boldsymbol{C} = \boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B}$
- Similar matrices have the same eigenvalues

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matlab>>roots(poly(eye(3)))'
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>>$(1 \quad 1 - 4.7606e-06 \cdot \boldsymbol{i} \quad 1 + 4.7606e-06 \cdot \boldsymbol{i})$

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- If $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ has distinct eigenvalues then $\boldsymbol{A}$ is diagonalizable
- Even for $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, eigenvalues might be complex
- Complex number $\boldsymbol{z} \in \mathbb{C}$ has real part x, imaginary part y
- Recall that for $\boldsymbol{u} \in \mathbb{R}^m$, $\|\boldsymbol{u}\|_2^2 = \boldsymbol{u}^T \boldsymbol{u}$. Extend to $\boldsymbol{u} \in \mathbb{C}^m$ by

$$\|\boldsymbol{u}\|_2^2 = (\bar{\boldsymbol{u}})^T \boldsymbol{u} = \boldsymbol{u}^* \boldsymbol{u}$$

- $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ is *unitarily diagonalizable* if there exists $\boldsymbol{Q} \in \mathbb{C}^{m \times m}$ such that

$$\boldsymbol{Q}\boldsymbol{Q}^* = \boldsymbol{Q}^*\boldsymbol{Q} = \boldsymbol{I}, \boldsymbol{A}\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{\Lambda} \Rightarrow \boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^*,$$

  with $\boldsymbol{\Lambda}$ diagonal eigenvalue matrix, $\boldsymbol{Q}$ unitary eigenvector matrix
- $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ is *orthogonally diagonalizable* if there exists $\boldsymbol{Q} \in \mathbb{R}^{m \times m}$ such that

$$\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}, \boldsymbol{A}\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{\Lambda} \Rightarrow \boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T,$$

  with $\boldsymbol{\Lambda}$ diagonal eigenvalue matrix, $\boldsymbol{Q}$ orthogonal eigenvector matrix.
- For $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, symmetric matrices ($\boldsymbol{A} = \boldsymbol{A}^T$), antisymmetric matrices ($\boldsymbol{A} = -\boldsymbol{A}^T$), normal matrices ($\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{A}^T\boldsymbol{A}$) are orthogonally diagonalizable.

- Suppose $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ diagonalizable, $\boldsymbol{A} = \boldsymbol{X} \boldsymbol{\Lambda} \boldsymbol{X}^{-1}$
- Repeated application of $\boldsymbol{A}$

$$\boldsymbol{A}^2 = (\boldsymbol{X} \boldsymbol{\Lambda} \boldsymbol{X}^{-1})(\boldsymbol{X} \boldsymbol{\Lambda} \boldsymbol{X}^{-1}) = \boldsymbol{X} \boldsymbol{\Lambda}^2 \boldsymbol{X}^{-1}$$

$$\boldsymbol{A}^k = (\boldsymbol{X} \boldsymbol{\Lambda} \boldsymbol{X}^{-1}) \cdot \dots \cdot (\boldsymbol{X} \boldsymbol{\Lambda} \boldsymbol{X}^{-1}) = \boldsymbol{X} \boldsymbol{\Lambda}^k \boldsymbol{X}^{-1}$$

- Above allows definition of $e^{\boldsymbol{A}}, \sin(\boldsymbol{A}), \cos(\boldsymbol{A})$, for example

$$e^x = \frac{1}{0!} x^0 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \dots + \frac{1}{k!} x^k + \dots \Rightarrow$$

$$e^{\boldsymbol{A}} = \boldsymbol{X} \left(\frac{1}{0!} \boldsymbol{\Lambda}^0 + \frac{1}{1!} \boldsymbol{\Lambda} + \frac{1}{2!} \boldsymbol{\Lambda}^2 + \dots + \frac{1}{k!} \boldsymbol{\Lambda}^k + \dots \right) \boldsymbol{X}^{-1}$$

- The differential system $\boldsymbol{y}' = \boldsymbol{A} \boldsymbol{y}$ has solution $\boldsymbol{y}(t) = e^{\boldsymbol{A} t} y(0)$.