MATH347.SP.01 Midterm Examination

Instructions. *Answer the following questions. Provide concise motivation of your approach. Illegible answers are not awarded any credit. Presentation of calculations without mention of the motivation and reasoning are not awarded any credit.*

1. Compute the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$, (2 points)

$$\boldsymbol{b} = \begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}, \boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 3 & -6 \\ 6 & 2 \end{bmatrix}.$$

Solution. $\boldsymbol{A} = [\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2\ ]$ has orthogonal column vectors, $\boldsymbol{a}_1^T \boldsymbol{a}_2 = 2 \cdot 3 + 3 \cdot (-6) + 6 \cdot 2 = 0$, so $\boldsymbol{A}$ has QR decomposition with

$$\boldsymbol{Q} = \left[\begin{array}{cc} \frac{\boldsymbol{a}_1}{\|\boldsymbol{a}_1\|} & \frac{\boldsymbol{a}_2}{\|\boldsymbol{a}_2\|} \end{array} \right],$$

with $\|\boldsymbol{a}_1\| = \|\boldsymbol{a}_2\| = (2^2 + 3^2 + 6^2)^{1/2} = (4 + 9 + 36)^{1/2} = \sqrt{49}$, hence

$$\boldsymbol{Q} = \frac{1}{\sqrt{49}} \boldsymbol{A}$$

The projector onto $C(\boldsymbol{A})$ is $\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{Q}^T$, and the projection of $\boldsymbol{b}$ onto $C(\boldsymbol{A})$ is

$$\boldsymbol{c} = \boldsymbol{P}\boldsymbol{b} = \boldsymbol{Q}\boldsymbol{Q}^T\boldsymbol{b} = \frac{1}{49}\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{b} = \begin{bmatrix} 2 & 3 \\ 3 & -6 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 6 \\ 3 & -6 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & -6 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 11 \\ -1 \end{bmatrix} = \begin{bmatrix} 19 \\ 39 \\ 64 \end{bmatrix}.$$

2. Let R: $\mathbb{R}^2 \to \mathbb{R}^2$ denote rotation by angle θ. Let S: $\mathbb{R}^2 \to \mathbb{R}^2$ denote a stretching transformation by α along the x_1 axis, and β along the x_2 axis. Let $T = R \circ S$ denote the composite transformation of stretching followed by rotation. (3 points)

a) Write the matrix $\boldsymbol{A}$ representing R.

Solution.
$$\boldsymbol{A} = [\ R(\boldsymbol{e}_1) \ \ R(\boldsymbol{e}_2)\ ] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

b) Write the matrix $\boldsymbol{B}$ representing S.

Solution.
$$\boldsymbol{B} = [\ S(\boldsymbol{e}_1) \ \ S(\boldsymbol{e}_2)\ ] = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}.$$

c) Write the matrix $\boldsymbol{C}$ representing T.

Solution.
$$\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} \alpha\cos\theta & -\beta\sin\theta \\ \alpha\sin\theta & \beta\cos\theta \end{bmatrix}.$$

3. Consider (5 points)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) What is the rank of $\boldsymbol{A}$?

Solution. $\boldsymbol{A} \in \mathbb{R}^{4\times 5}$, $m = 4, n = 5$ is already in rref. Rank is number of pivot rows (columns), $\operatorname{rank}(\boldsymbol{A}) = 3$.

b) State a basis for $C(\boldsymbol{A})$.

Solution. $C(\boldsymbol{A}) = \{\boldsymbol{b} \mid \exists \boldsymbol{x}\, s.t. \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$. Choose the columns with pivot elements

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

c) State a basis for $N(\boldsymbol{A}^T)$.

Solution. $N(\boldsymbol{A}^T) = \{\boldsymbol{y} \mid \boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}\}$. Since $C(\boldsymbol{A}) \oplus N(\boldsymbol{A}^T) = \mathbb{R}^4$, choose $\boldsymbol{e}_4$

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

d) State a basis for $C(\boldsymbol{A}^T)$.

Solution. Choose rows with pivot elements

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 5 \end{bmatrix} \right\}.$$

e) State a basis for $N(\boldsymbol{A})$.

Solution. FTLA states $C(\boldsymbol{A}^T) \oplus N(\boldsymbol{A}) = \mathbb{R}^n$, and since $\dim C(\boldsymbol{A}^T) = \operatorname{rank}(\boldsymbol{A}) = 3$, it results that $\dim N(\boldsymbol{A}) = n - 3 = 5 - 3 = 2$, hence two basis vectors are required. From system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ obtain

$$\begin{cases} x_1 = -2x_2 - 3x_5 \\ x_3 = -4x_5 \\ x_4 = -5x_5 \end{cases},$$

in which x_2, x_5 are free parameters. For $x_2 = 1$, $x_5 = 0$ obtain first basis vector

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and for $x_2 = 0, x_5 = 1$ obtain second basis vector

$$\begin{bmatrix} -3 \\ 0 \\ -4 \\ -5 \\ 1 \end{bmatrix}.$$