MATH347.SP.25 Midterm Examination

Instructions. *Answer the following questions. Provide concise motivation of your approach. Illegible answers are not awarded any credit. Presentation of calculations without mention of the motivation and reasoning are not awarded any credit. Each correct question answer is awarded 2 course points.*

1. Prove the trigonometric identity

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta).$$

Notation: $\cos^2(\theta) = [\cos(\theta)]^2$, $\sin^2(\theta) = [\sin(\theta)]^2$.

Solution. The matrix

$$\boldsymbol{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

describes rotation by angle θ in $\mathbb{R}^2$. Rotation by angle 3θ is obtained by repeated application

$$\boldsymbol{R}_\theta^3 = \boldsymbol{R}_{3\theta} \Rightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 3\theta & -\sin 3\theta \\ \sin 3\theta & \cos 3\theta \end{bmatrix}$$

Carrying out the multiplications gives

$$\boldsymbol{R}_\theta^3 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin\theta \end{bmatrix} = \begin{bmatrix} \cos^3\theta - \cos\theta\sin^2\theta - 2\sin^2\theta\cos\theta & - \\ - & - \end{bmatrix}$$

and equality of the 1,1 component gives

$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta,$$

as requested.

2. Determine the standard matrix $\boldsymbol{P}$ of the orthogonal projection of a vector $\boldsymbol{v} \in \mathbb{R}^3$ onto the line $x_1 = x_2 = x_3$.

Solution. The unit vector along the line $x_1 = x_2 = x_3$ is

$$\boldsymbol{q} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and the projection matrix along this direction is

$$\boldsymbol{P} = \boldsymbol{q}\boldsymbol{q}^T = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

3. Determine bases for the fundamental subspaces of the matrix $\boldsymbol{P}$ defined above.

Solution. For $\boldsymbol{u}\in\mathbb{R}^3$, $\boldsymbol{v}=\boldsymbol{P}\boldsymbol{u}=(\boldsymbol{q}\boldsymbol{q}^T)\boldsymbol{u}=(\boldsymbol{q}^T\boldsymbol{u})\boldsymbol{q}$ is in the direction of $\boldsymbol{q}$ hence a basis for $C(\boldsymbol{P})$ is $\{\boldsymbol{q}\}$and $r=\text{rank}(\boldsymbol{P})=1$. The left null space contains vectors orthogonal to $\boldsymbol{q}$, for example

$$\boldsymbol{y}=\begin{bmatrix}1\\-1\\0\end{bmatrix},\boldsymbol{z}=\begin{bmatrix}1\\0\\-1\end{bmatrix},$$

that verify $\boldsymbol{y}^T\boldsymbol{q}=0$, $\boldsymbol{z}^T\boldsymbol{q}=0$, and linearly independent, hence $\{\boldsymbol{y},\boldsymbol{z}\}$ is a basis. Note that $\boldsymbol{P}^T=\boldsymbol{P}$ such that $\{\boldsymbol{q}\}$ is a basis for the row space $C(\boldsymbol{P}^T)$, and $\{\boldsymbol{y},\boldsymbol{z}\}$ is a basis for $N(\boldsymbol{P})$.

4. Find the inverse of the standard matrix $\boldsymbol{M}\in\mathbb{R}^{3\times 3}$ of the linear mapping, $L=F\circ G$ with

   a) $F:\mathbb{R}^3\rightarrow\mathbb{R}^3$ denoting reflection across the vector $\boldsymbol{u}=[\,1\ \ 1\ \ 0\,]^T$;

   b) $G:\mathbb{R}^3\rightarrow\mathbb{R}^3$ denoting scaling by $\lambda_1,\lambda_2,\lambda_3$ along directions x_1,x_2,x_3, respectively.

Solution. Consider $\boldsymbol{v},\boldsymbol{w},\boldsymbol{z}\in\mathbb{R}^3$ and diagram in Fig. 1, with $\boldsymbol{w}$ the projection of $\boldsymbol{v}$ onto the direction of $\boldsymbol{u}$. Let

$$\boldsymbol{q}=\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}.$$

The projection matrix is then

$$\boldsymbol{P}=\boldsymbol{q}\boldsymbol{q}^T=\frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}[\,1\ \ 1\ \ 0\,]=\frac{1}{2}\begin{bmatrix}1&1&0\\1&1&0\\0&0&0\end{bmatrix},$$

leading to $\boldsymbol{w}=\boldsymbol{P}\boldsymbol{v}$. The vector $\boldsymbol{w}$ is also the sum

$$\boldsymbol{w}=\boldsymbol{v}+\boldsymbol{z}.$$

The reflection of $\boldsymbol{v}$ across $\boldsymbol{u}$ is reached by traveling from endpoint of $\boldsymbol{v}$ by $2\boldsymbol{z}$

$$\boldsymbol{y}=\boldsymbol{w}+2\boldsymbol{z}=\boldsymbol{w}+2(\boldsymbol{w}-\boldsymbol{v})=2\boldsymbol{w}-\boldsymbol{v}=2\boldsymbol{P}\boldsymbol{v}-\boldsymbol{v}=(2\boldsymbol{P}-\boldsymbol{I})\boldsymbol{v}.$$

From the above deduce the standard matrix for reflection across $\boldsymbol{u}$ is

$$\boldsymbol{A}=2\boldsymbol{P}-\boldsymbol{I}=\begin{bmatrix}1&1&0\\1&1&0\\0&0&0\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}.$$

The matrix for scaling is

$$\boldsymbol{B}=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}.$$

The matrix for the composite transformation is

$$\boldsymbol{M}=\boldsymbol{A}\boldsymbol{B},$$

with inverse

$$M^{-1} = B^{-1} A^{-1}.$$

The inverse of $\boldsymbol{B}$ is

$$B^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 & 0 \\ 0 & 1/\lambda_2 & 0 \\ 0 & 0 & 1/\lambda_3 \end{bmatrix},$$

assuming non-zero λ_1, λ_2, λ_3. Note that reflection of $\boldsymbol{y}$ across $\boldsymbol{u}$ gives the original vector $\boldsymbol{v}$. Stated in matrix terms

$$A A = I,$$

$\boldsymbol{A}$ is its own inverse. Then

$$M = B^{-1}(2P - I) = \begin{bmatrix} 1/\lambda_1 & 0 & 0 \\ 0 & 1/\lambda_2 & 0 \\ 0 & 0 & 1/\lambda_3 \end{bmatrix} \left(2 \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \Rightarrow$$

$$M^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 & 0 \\ 0 & 1/\lambda_2 & 0 \\ 0 & 0 & 1/\lambda_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1/\lambda_1 & 0 \\ 1/\lambda_2 & 0 & 0 \\ 0 & 0 & -1/\lambda_3 \end{bmatrix}.$$

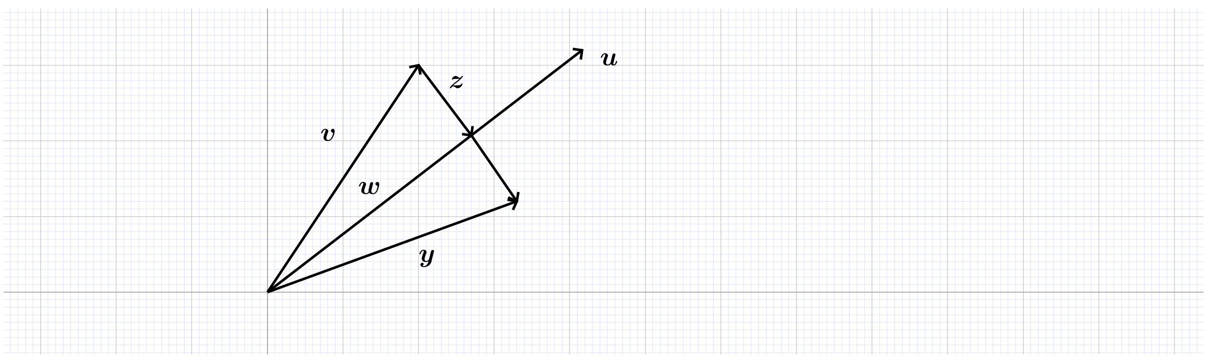

Figure 1.

5. Compute the LU factorization without permutations of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 2 \\ 0 & -8 & -4 \end{bmatrix}.$$

Explicitly state the elementary matrices used at each stage of the process.

Solution. The stage 1 operation is

$$L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 2 \\ 0 & -8 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & -8 & -4 \end{bmatrix}.$$

The stage 2 operation is

$$\boldsymbol{L}_2(\boldsymbol{L}_1\boldsymbol{A}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & -8 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = \boldsymbol{U}.$$

Multiply by inverse to obtain

$$\boldsymbol{A} = \boldsymbol{L}_1^{-1}\boldsymbol{L}_2^{-1}\boldsymbol{U} = \boldsymbol{L}\boldsymbol{U},$$

with

$$\boldsymbol{L} = \boldsymbol{L}_1^{-1}\boldsymbol{L}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$