Practice Final Examination

Solve the following problems (5 course points each). Present a brief motivation of your method of solution. Problems 9 and 10 are optional; attempt them if you wish to improve your midterm examination score.

1. State the matrix product to obtain 3 linear combinations of vectors

$$\boldsymbol{u}=\begin{bmatrix}1\\0\\1\end{bmatrix}, \boldsymbol{v}=\begin{bmatrix}-1\\0\\1\end{bmatrix},$$

with scaling coefficients $(\alpha_1,\beta_1)=(1,1)$, $(\alpha_2,\beta_2)=(-1,1)$, $(\alpha_3,\beta_3)=(1,-1)$.

Solution. The matrix product is $\boldsymbol{C}=\boldsymbol{A}\boldsymbol{B}$, $\boldsymbol{C}\in\mathbb{R}^{3\times 3}$ (3 linear combinations, each with 3 components), $\boldsymbol{A}=[\begin{array}{cc}\boldsymbol{u} & \boldsymbol{v}\end{array}]\in\mathbb{R}^{3\times 2}$ (vectors entering into linear combination), $\boldsymbol{B}\in\mathbb{R}^{2\times 3}$ (scaling coefficients of each linear combination)

$$\boldsymbol{A}=\begin{bmatrix}1 & -1\\0 & 0\\1 & 1\end{bmatrix}, \boldsymbol{B}=\begin{bmatrix}1 & -1 & 1\\1 & 1 & -1\end{bmatrix}.$$

2. Orthonormalize the vectors

$$\boldsymbol{u}=\begin{bmatrix}1\\0\\1\end{bmatrix}, \boldsymbol{v}=\begin{bmatrix}-1\\0\\1\end{bmatrix}, \boldsymbol{w}=\begin{bmatrix}1\\1\\-1\end{bmatrix}.$$

Solution. Note that $\boldsymbol{u}^T\boldsymbol{v}=\boldsymbol{u}^T\boldsymbol{w}=0$, such that these vector pairs are already orthogonal. Also note that the vector $\boldsymbol{e}_2=[\begin{array}{ccc}0 & 1 & 0\end{array}]^T$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$. Scale vectors to have unit norm

$$\boldsymbol{Q}=[\begin{array}{ccc}\boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3\end{array}]=[\begin{array}{ccc}\boldsymbol{u}/\sqrt{2} & \boldsymbol{v}/\sqrt{2} & \boldsymbol{e}_2\end{array}].$$

3. For $x,y\in\mathbb{R}$, expansion of $(x-y)^3$ leads to $(x-y)^3=x^3-3x^2y+3xy^2-y^3$. Find the corresponding expansion of $(\boldsymbol{A}-\boldsymbol{B})^3$ for $\boldsymbol{A},\boldsymbol{B}\in\mathbb{R}^{m\times m}$.

Solution. Compute, recalling that matrix multiplication is not commutative

$$(\boldsymbol{A}-\boldsymbol{B})^3=(\boldsymbol{A}-\boldsymbol{B})(\boldsymbol{A}^2-\boldsymbol{A}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{A}+\boldsymbol{B}^2)=\boldsymbol{A}^3-\boldsymbol{A}^2\boldsymbol{B}-\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}+\boldsymbol{A}\boldsymbol{B}^2-\boldsymbol{B}\boldsymbol{A}^2+\boldsymbol{B}\boldsymbol{A}\boldsymbol{B}+\boldsymbol{B}^2-\boldsymbol{B}^3.$$

4. Find the projection of $\boldsymbol{b}$ onto $C(\boldsymbol{A})$ for

$$\boldsymbol{b}=\begin{bmatrix}1\\2\\3\end{bmatrix}, \boldsymbol{A}=\begin{bmatrix}1 & -1\\1 & 0\\1 & 1\end{bmatrix}.$$

Solution. With $\boldsymbol{A}=[\begin{array}{cc}\boldsymbol{a}_1 & \boldsymbol{a}_2\end{array}]$ note that $\boldsymbol{b}=2\boldsymbol{a}_1+\boldsymbol{a}_2$, hence $\boldsymbol{b}\in C(\boldsymbol{A})$, and the projection of $\boldsymbol{b}$ onto $C(\boldsymbol{A})$ is $\boldsymbol{b}$ itself.

Alternatively, orthonormalize $\boldsymbol{A}=[\begin{array}{cc}\boldsymbol{a}_1 & \boldsymbol{a}_2\end{array}]$ to obtain

$$\boldsymbol{Q}=[\begin{array}{cc}\boldsymbol{q}_1 & \boldsymbol{q}_2\end{array}]=[\begin{array}{cc}\boldsymbol{a}_1/\sqrt{3} & \boldsymbol{a}_2/\sqrt{2}\end{array}].$$

The projection onto $C(\boldsymbol{A})$ is

$$\boldsymbol{c}=\boldsymbol{Q}\boldsymbol{Q}^T\boldsymbol{b}=\boldsymbol{Q}\,(\boldsymbol{Q}^T\boldsymbol{b})=\begin{bmatrix}1/\sqrt{3} & -1/\sqrt{2}\\1/\sqrt{3} & 0\\1/\sqrt{3} & 1/\sqrt{2}\end{bmatrix}\left(\begin{bmatrix}1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}\\-1/\sqrt{2} & 0 & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\right)\Rightarrow$$

$$\boldsymbol{c}=\begin{bmatrix}1/\sqrt{3} & -1/\sqrt{2}\\1/\sqrt{3} & 0\\1/\sqrt{3} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}2\sqrt{3}\\\sqrt{2}\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}.$$

5. Find the LU decomposition of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{bmatrix}.$$

Solution. Carry out reduction to upper triangular form, noting multipliers used in the process

$$\boldsymbol{L}_1 \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}.$$

$$\boldsymbol{L}_2 \boldsymbol{L}_1 \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{U}.$$

Find $\boldsymbol{A} = \boldsymbol{L}_1^{-1} \boldsymbol{L}_2^{-1} \boldsymbol{U} = \boldsymbol{L}\boldsymbol{U}$. Compute

$$\boldsymbol{L} = \boldsymbol{L}_1^{-1} \boldsymbol{L}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

Verify

$$\boldsymbol{L}\boldsymbol{U} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{bmatrix}. \checkmark$$

6. State the eigenvalues and eigenvectors of $\boldsymbol{R} \in \mathbb{R}^{2\times 2}$, the matrix describing reflection across the vector $\boldsymbol{w} = [\,1 \;\; 2\,]^T$.

Solution. From eigenvalue relation $\boldsymbol{R}\boldsymbol{x} = \lambda \boldsymbol{x}$ note that directions not changed by reflection are along $\boldsymbol{w}$ and orthogonal to $\boldsymbol{w}$. Deduce

$$\boldsymbol{x}_1 = \boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \lambda_1 = 1$$

$$\boldsymbol{x}_2 = \boldsymbol{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \lambda_2 = -1.$$

7. Compute the eigendecomposition of

$$\boldsymbol{A} = \begin{bmatrix} 5/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 5/2 \end{bmatrix}.$$

Solution. Eigenvalue problem is $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$. Observe that $\boldsymbol{x}_2 = \boldsymbol{e}_2 = [\,0 \;\; 1 \;\; 0\,]^T$, $\lambda_2 = 1$ is an eigenvector, value pair. Compute characteristic polynomial by

$$p(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda - 5/2 & 0 & -1/2 \\ 0 & \lambda - 1 & 0 \\ -1/2 & 0 & \lambda - 5/2 \end{vmatrix} = (\lambda - 1) \begin{vmatrix} \lambda - 5/2 & -1/2 \\ -1/2 & \lambda - 5/2 \end{vmatrix} \Rightarrow$$

$$p(\lambda) = (\lambda - 1)\left(\lambda^2 - 5\lambda + \left(\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \right) = (\lambda - 1)(\lambda^2 - 5\lambda + 6) = (\lambda - 1)(\lambda - 2)(\lambda - 3).$$

The other eigevalues are $\lambda_1 = 2$, $\lambda_3 = 3$. Find eigenvectors by computing bases for eigenspaces $N(\boldsymbol{A} - \lambda_1 \boldsymbol{I})$ and $N(\boldsymbol{A} - \lambda_3 \boldsymbol{I})$.

$$\boldsymbol{A} - \lambda_1 \boldsymbol{I} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \sim \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\boldsymbol{A} - \lambda_3 \boldsymbol{I} = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix} \sim \begin{bmatrix} -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

8. Find the SVD of

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Solution. Matrix has orthogonal columns that are not of unit norm. Construct SVD as

$$\boldsymbol{A} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

9. Find the matrix of the reflection of $\mathbb{R}^2$ vectors across the vector $\boldsymbol{u} = [\,1 \;\; 2\,]^T$.

Solution. Let $\boldsymbol{w}$ be the reflection of $\boldsymbol{v}$ across $\boldsymbol{u}$, $\boldsymbol{w} = \boldsymbol{R}\boldsymbol{v}$. Let

$$\boldsymbol{q} = \frac{1}{\|\boldsymbol{u}\|}\boldsymbol{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The projection of $\boldsymbol{v}$ onto the direction of $\boldsymbol{u}$ is $\boldsymbol{z} = \boldsymbol{q}\boldsymbol{q}^T\boldsymbol{v}$. The travel from $\boldsymbol{v}$ to $\boldsymbol{z}$ is $\boldsymbol{z} - \boldsymbol{v}$

$$\boldsymbol{z} = \boldsymbol{v} + (\boldsymbol{z} - \boldsymbol{v}).$$

The reflection is obtained by doubling the travel distance

$$\boldsymbol{w} = \boldsymbol{v} + 2(\boldsymbol{z} - \boldsymbol{v}) = 2\boldsymbol{z} - \boldsymbol{v} = (2\boldsymbol{q}\boldsymbol{q}^T - \boldsymbol{I})\boldsymbol{v}.$$

Deduce that the reflection matrix is

$$\boldsymbol{R} = 2\boldsymbol{q}\boldsymbol{q}^T - \boldsymbol{I}.$$

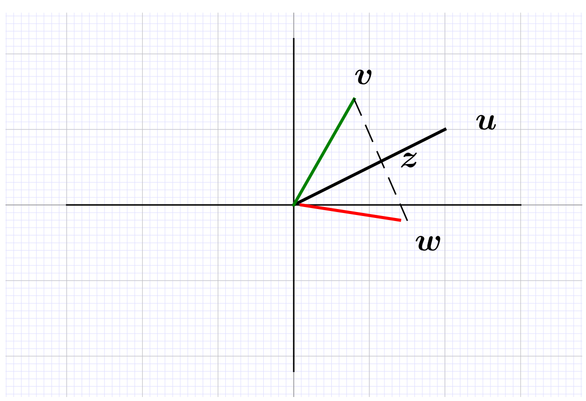

Figure 1.

10. Find bases for the four fundamental spaces of

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Solution. With $\boldsymbol{A} = [\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ ] \in \mathbb{R}^{3\times 2}$, the bases are:

$C(\boldsymbol{A})$: $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$

$N(\boldsymbol{A}^T)$: $\{\boldsymbol{e}_2\}$ since $\boldsymbol{e}_2$ is orthogonal to both $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$

$C(\boldsymbol{A}^T)$: $\{\boldsymbol{a}_1^T, \boldsymbol{a}_2^T\}$

$N(\boldsymbol{A})$: $\{\boldsymbol{0}\}$