MATH347.SP.25 Midterm Solution

Instructions. *Answer the following questions. Provide concise motivation of your approach. Illegible answers are not awarded any credit. Presentation of calculations without mention of the motivation and reasoning are not awarded any credit. Each correct question answer is awarded 2 course points.*

Note. *Solution briefly states motivation/approach and then concisely carries out required calculations.*

1. Find the image of the standard basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \in \mathbb{R}^3$ through the linear mapping $W: \mathbb{R}^3 \to \mathbb{R}^3$ defined by the composition $W = T \circ S$, where:

a) S is rotation around the x_3 axis by $\theta = \pi/4$;

b) T is rotation around the x_2 axis by $\theta = \pi/4$.

Solution. Denote by $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{3\times 3}$ the standard matrices of mappings S, T, W, respectively, such that

$$S(\boldsymbol{u}) = \boldsymbol{A}u, T(\boldsymbol{v}) = \boldsymbol{B}\boldsymbol{v}, W(\boldsymbol{z}) = \boldsymbol{C}\boldsymbol{z}.$$

Since $W = T \circ S$, $\boldsymbol{C} = \boldsymbol{B}\boldsymbol{A}$.

a) Rotation around the x_3 axis by $\theta = \pi/4$ has standard matrix

$$\boldsymbol{A} = [\ S(\boldsymbol{e}_1) \ \ S(\boldsymbol{e}_2) \ \ S(\boldsymbol{e}_3)\ ] = \left[\begin{array}{lll} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{lll} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{array}\right].$$

b) Rotation around the x_2 axis by $\theta = \pi/4$ has standard matrix

$$\boldsymbol{B} = [\ T(\boldsymbol{e}_1) \ \ T(\boldsymbol{e}_2) \ \ T(\boldsymbol{e}_3)\ ] = \left[\begin{array}{lll} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{array}\right] = \left[\begin{array}{lll} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{array}\right].$$

Therefore,

$$\boldsymbol{C} = \left[\begin{array}{lll} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{array}\right] \left[\begin{array}{lll} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{lll} 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} \end{array}\right].$$

Since $W(\boldsymbol{e}_j) = \boldsymbol{C}\boldsymbol{e}_j$, obtain $\boldsymbol{C} = [\ \boldsymbol{c}_1 \ \ \boldsymbol{c}_2 \ \ \boldsymbol{c}_3\ ] = [\ W(\boldsymbol{e}_1) \ \ W(\boldsymbol{e}_2) \ \ W(\boldsymbol{e}_3)\ ]$.

2. Let $\boldsymbol{C}$ denote the standard matrix of the linear mapping W defined above. State bases for the four fundamental subspaces of $\boldsymbol{C}$.

Solution. Rotation of the orthonormal vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ by the same angle yields three orthonormal vectors $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$. Verification

$$\boldsymbol{C}^T\boldsymbol{C} = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ -1/2 & 1/\sqrt{2} & -1/2 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} \end{bmatrix} = \boldsymbol{I}.$$

Since columns of $\boldsymbol{C} = [\ \boldsymbol{c}_1\ \ \boldsymbol{c}_2\ \ \boldsymbol{c}_3\ ]$ are orthonormal, they are linearly independent, and $\text{rank}(\boldsymbol{C}) = 3$. Any three linearly independent vectors form a basis for both column space and row space, for instance, $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. The null spaces $N(\boldsymbol{C}), N(\boldsymbol{C}^T)$ are of dimension zero, and do not have a basis.

3. Determine an orthonormal basis for $C(\boldsymbol{C})$ with $\boldsymbol{C}$ defined above.

Solution. As noted above, $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is an orthonormal basis for $C(\boldsymbol{C})$. Another basis is $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$.

4. Consider the vector $\boldsymbol{b} = [\ 1\ \ 1\ \ 1\ ]^T$. Find the coordinates of $\boldsymbol{b}$ in the basis defined by the column vectors of $\boldsymbol{C}$ defined above.

Solution. In the $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ basis the coordinates of $\boldsymbol{b}$ are 1,1,1. In the $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ basis, the coordinates are

$$\boldsymbol{x} = \boldsymbol{C}^T\boldsymbol{b} = \begin{bmatrix} 1 + \frac{1}{\sqrt{2}} \\ -1 + \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}.$$

5. Compute the inverse of the matrix

$$\boldsymbol{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} - \frac{1}{2\sqrt{2}} & \frac{1}{2} - \frac{1}{2\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2\sqrt{2}} & -\frac{1}{2} - \frac{1}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix}.$$

Solution. Check if the matrix has orthogonal columns. Computing

$$\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I},$$

confirming that columns of $\boldsymbol{A}$ are orthonormal, hence $\boldsymbol{A}^{-1} = \boldsymbol{A}^T$.