MATH347: Linear algebra for applications

Homework 10 - Solution

This assignment is a worksheet of exercises intended as preparation for the Final Examination. You should:

1. Review Lessons 1 to 12
2. Set aside 60 minutes to solve these exercises. Each exercise is meant to be solved within 3 minutes. If you cannot find a solution within 3 minutes, skip to the next one.
3. Check your answers in Matlab. Revisit theory for skipped or incorrectly answered exercies.
4. Turn in a PDF with your brief handwritten answers that specify your motivation, approach, calculations, answer. It is good practice to start all answers by briefly recounting the applicable definitions.

When constructing a solution follow these steps:

a) Ask yourself: "what course concept is being verified?"
b) Identify relevant definitions and include them in your answer.
c) Briefly describe your approach
d) Carry out calculations
e) Present final answer

1 Vector operations

1. Find the linear combination of vectors $\boldsymbol{u} = [\,1\ \ 1\ \ 1\,]$, $\boldsymbol{v} = [\,1\ \ 2\ \ 3\,]$ with scaling coefficients $\alpha = 2$, $\beta = 1$.

   Solution. By definition of linear combination
$$\boldsymbol{w} = \alpha\boldsymbol{u} + \beta\boldsymbol{v} = 2\cdot[\,1\ \ 1\ \ 1\,] + 1\cdot[\,1\ \ 2\ \ 3\,] = [\,3\ \ 4\ \ 5\,].$$

2. Express the above linear combination $\boldsymbol{b}$ as a matrix-vector product $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$. Define $\boldsymbol{x}$ and the column vectors of $\boldsymbol{A} = [\,\boldsymbol{a}_1\ \ \boldsymbol{a}_2\,]$.

   Solution. Vectors that enter the linear combination are matrix $\boldsymbol{A}$ columns, scaling coefficients are components of $\boldsymbol{x}$
$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} = [\,\boldsymbol{a}_1\ \ \boldsymbol{a}_2\,]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$$

3. Consider $\boldsymbol{u} = [\,1\ \ 1\ \ 0\,]^T$, $\boldsymbol{v} = [\,1\ \ 1\ \ 1\,]^T$. Compute the 2-norms of $\boldsymbol{u}, \boldsymbol{v}$. Determine the angle between $\boldsymbol{u}, \boldsymbol{v}$.

   Solution. By definition the 2-norms are
$$\|\boldsymbol{u}\|_2 = \left(\sum_{i=1}^{3} u_i^2\right)^{1/2} = \sqrt{2},\ \|\boldsymbol{v}\|_2 = \left(\sum_{i=1}^{3} v_i^2\right)^{1/2} = \sqrt{3}.$$

   The cosine of the angle θ between $\boldsymbol{u}, \boldsymbol{v}$ is defined as
$$\cos\theta = \frac{\boldsymbol{u}^T\boldsymbol{v}}{\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2} = \frac{2}{\sqrt{6}}.$$

4. Consider $\boldsymbol{u} = [\,1\ \ 1\ \ 0\,]^T$, $\boldsymbol{v} = [\,1\ \ 1\ \ 1\,]^T$. Define vector $\boldsymbol{w}$ such that $\boldsymbol{v} + \boldsymbol{w}$ is orthogonal to $\boldsymbol{u}$. Write the equation to determine $\boldsymbol{w}$, and then compute $\boldsymbol{w}$.

   Solution. Orthogonal vectors satisfy
$$\boldsymbol{u}^T(\boldsymbol{v} + \boldsymbol{w}) = 0,$$

whence

$$\boldsymbol{u}^T\boldsymbol{w}=[\ 1\ \ 1\ \ 0\ ]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}=-\boldsymbol{u}^T\boldsymbol{v}=-[\ 1\ \ 1\ \ 0\ ]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}=-2\Rightarrow w_1+w_2=-2.$$

The above equation has an infinity of solutions, say

$$\boldsymbol{w}=\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}.$$

Verify:

$$\boldsymbol{v}+\boldsymbol{w}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}+\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{u}^T(\boldsymbol{v}+\boldsymbol{w})=[\ 1\ \ 1\ \ 0\ ]\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}=0.\checkmark$$

5. Determine $\boldsymbol{q}_1, \boldsymbol{q}_2$ to be of unit norm and in the direction of vectors $\boldsymbol{u}, \boldsymbol{v}+\boldsymbol{w}$ from Ex. 4. Form $\hat{\boldsymbol{Q}}=[\ \boldsymbol{q}_1\ \ \boldsymbol{q}_2\ ]$. Compute $\hat{\boldsymbol{Q}}\hat{\boldsymbol{Q}}^T$ and $\hat{\boldsymbol{Q}}^T\hat{\boldsymbol{Q}}$.

   Solution. By definition of a unit norm vector

$$\boldsymbol{q}_1=\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|_2}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{q}_2=\frac{\boldsymbol{v}+\boldsymbol{w}}{\|\boldsymbol{v}+\boldsymbol{w}\|_2}=\frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

   leading to

$$\hat{\boldsymbol{Q}}=\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{bmatrix}.$$

   Compute products

$$\hat{\boldsymbol{Q}}\hat{\boldsymbol{Q}}^T=\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}=\begin{bmatrix} 5/6 & 1/6 & -1/3 \\ 1/6 & 5/6 & 1/3 \\ -1/3 & 1/3 & 1/3 \end{bmatrix},$$

   a projection matrix onto $C(\hat{\boldsymbol{Q}})$

$$\hat{\boldsymbol{Q}}^T\hat{\boldsymbol{Q}}=\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

   the identity matrix.

6. Determine vector $\boldsymbol{q}_3$ orthonormal to vectors $\boldsymbol{q}_1, \boldsymbol{q}_2$ from Ex. 5.

   Solution. By observation, choose

$$\boldsymbol{q}_3=\frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}.$$

   Verify

$$\boldsymbol{q}_1^T\boldsymbol{q}_3=\frac{1}{\sqrt{12}}[\ 1\ \ 1\ \ 0\ ]\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}=0, \boldsymbol{q}_2^T\boldsymbol{q}_3=\frac{1}{\sqrt{18}}[\ -1\ \ 1\ \ 1\ ]\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}=0, \boldsymbol{q}_3^T\boldsymbol{q}_3=\frac{1}{6}[\ -1\ \ 1\ \ -2\ ]\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}=1$$

7. Establish whether vectors $\boldsymbol{u}=[\ 1\ \ 2\ \ 3\ ]^T$, $\boldsymbol{v}=[\ -3\ \ 1\ \ -2\ ]^T$, $\boldsymbol{w}=[\ 2\ \ -3\ \ 1\ ]^T$ all lie in the same plane within $\mathbb{R}^3$.

Solution. The vectors would have to be linearly dependent. Form the matrix

$$\boldsymbol{A}=[\,\boldsymbol{u}\;\;\boldsymbol{v}\;\;\boldsymbol{w}\,]=\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 3 & -2 & 1 \end{bmatrix}.$$

Carry out reduction to rref

$$\boldsymbol{A}\sim\begin{bmatrix} 1 & -3 & 2 \\ 0 & 7 & -7 \\ 0 & 7 & -5 \end{bmatrix}\sim\begin{bmatrix} 1 & -3 & 2 \\ 0 & 7 & -7 \\ 0 & 0 & 2 \end{bmatrix},$$

a matrix of full rank, hence with linearly independent columns, implying that $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ do not all lie in the same plane.

8. Determine $\boldsymbol{v}$ the reflection of vector $\boldsymbol{u}=[\,1\;\;\sqrt{3}\,]^T$ across vector $\boldsymbol{w}=[\,1\;\;1\,]^T$.

   Solution. Form unit vector

   $$\boldsymbol{q}=\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   The reflection matrix is

   $$\boldsymbol{R}=2\boldsymbol{q}\boldsymbol{q}^T-\boldsymbol{I}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}[\,1\;\;1\,]-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

   Compute reflection

   $$\boldsymbol{R}\boldsymbol{u}=\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}.$$

9. Determine $\boldsymbol{w}$ the rotation of vector $\boldsymbol{u}=[\,1\;\;\sqrt{3}\,]^T$ by angle $\theta=-\pi/6$.

   Solution. The rotation matrix is

   $$\boldsymbol{R}=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}=\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix},$$

   and the rotated vector is

   $$\boldsymbol{w}=\boldsymbol{R}\boldsymbol{u}=\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}=\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}.$$

10. Compute $\boldsymbol{z}=\boldsymbol{v}-\boldsymbol{w}$ with $\boldsymbol{v},\boldsymbol{w}$ from Ex. 8,9.

    Solution. The difference is $\boldsymbol{z}=\boldsymbol{v}-\boldsymbol{w}$ highlighting that rotation and reflection of a vector can produce the same result.

2 Matrix operations

1. Find two linear combinations of vectors $\boldsymbol{u}=[\,1\;\;1\;\;1\,]$, $\boldsymbol{v}=[\,1\;\;2\;\;3\,]$ first with scaling coefficients $\alpha=2$, $\beta=1$, and then with scaling coefficients $\alpha=1$, $\beta=2$.

   Solution. Organize the multiple linear combinations as a matrix-matrix product

   $$\boldsymbol{A}\boldsymbol{B}=[\,\boldsymbol{u}\;\;\boldsymbol{v}\,]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 3 & 3 \\ 4 & 4 \\ 5 & 7 \end{bmatrix}.$$

2. Express the above linear combinations $\boldsymbol{B}$ as a matrix-matrix product $\boldsymbol{B}=\boldsymbol{A}\boldsymbol{X}$. Define the column vectors of $\boldsymbol{A},\boldsymbol{X}$.

   Solution. As above.

3. Consider $\boldsymbol{A},\boldsymbol{B}\in\mathbb{R}^{m\times m}$. Which of the following matrices are always equal to $\boldsymbol{C}=(\boldsymbol{A}-\boldsymbol{B})^2$?

   a) $\boldsymbol{A}^2-\boldsymbol{B}^2$

b) $(\boldsymbol{B}-\boldsymbol{A})^2$

c) $\boldsymbol{A}^2-2\boldsymbol{A}\boldsymbol{B}+\boldsymbol{B}^2$

d) $\boldsymbol{A}(\boldsymbol{A}-\boldsymbol{B})-\boldsymbol{B}(\boldsymbol{B}-\boldsymbol{A})$

e) $\boldsymbol{A}^2-\boldsymbol{A}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{A}+\boldsymbol{B}^2$

Solution. Expand the product taking into account that matrix multiplication is not commutative

$$\boldsymbol{C}=(\boldsymbol{A}-\boldsymbol{B})(\boldsymbol{A}-\boldsymbol{B})=\boldsymbol{A}^2-\boldsymbol{B}\boldsymbol{A}-\boldsymbol{A}\boldsymbol{B}+\boldsymbol{B}^2.$$

Of the possible choices only (b,e) are always equal to $\boldsymbol{C}$.

4. Find the inverse of

$$\boldsymbol{A}=\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Solution. Apply Gauss-Jordan algorithm

$$[\,\boldsymbol{A}\ \ \boldsymbol{I}\,]=\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}\sim\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}\sim$$

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 0 & 1 \end{bmatrix}\sim\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 0 & 1 \end{bmatrix}\sim$$

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 4/3 & -1/3 & -1/3 & 1 \end{bmatrix}\sim\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{bmatrix}\sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 3/4 & -1/4 & -1/4 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{bmatrix}\Rightarrow\boldsymbol{A}^{-1}=\begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}.$$

Alternatively, notice that

$$\boldsymbol{A}=\boldsymbol{I}-\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}[\,1\ \ 1\ \ 1\,],$$

and the formula in Ex. 5 below gives

$$\boldsymbol{A}^{-1}=\boldsymbol{I}+\frac{\boldsymbol{u}\boldsymbol{v}^T}{1-\boldsymbol{v}^T\boldsymbol{u}}=\boldsymbol{I}+\frac{1}{4}[\,-1\ \ -1\ \ -1\,]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}.$$

5. Verify that the inverse of $\boldsymbol{A}=\boldsymbol{I}-\boldsymbol{u}\boldsymbol{v}^T$ is

$$\boldsymbol{A}^{-1}=\boldsymbol{I}+\frac{\boldsymbol{u}\boldsymbol{v}^T}{1-\boldsymbol{v}^T\boldsymbol{u}}$$

when $\boldsymbol{v}^T\boldsymbol{u}\neq 1$.

Solution. Verify that $\boldsymbol{A}\boldsymbol{A}^{-1}=\boldsymbol{A}^{-1}\boldsymbol{A}=\boldsymbol{I}$.

$$\boldsymbol{A}\boldsymbol{A}^{-1}=(\boldsymbol{I}-\boldsymbol{u}\boldsymbol{v}^T)\left(\boldsymbol{I}+\frac{\boldsymbol{u}\boldsymbol{v}^T}{1-\boldsymbol{v}^T\boldsymbol{u}}\right)=\boldsymbol{I}-\boldsymbol{u}\boldsymbol{v}^T+\frac{\boldsymbol{u}\boldsymbol{v}^T}{1-\boldsymbol{v}^T\boldsymbol{u}}-\frac{\boldsymbol{u}\,(\boldsymbol{v}^T\boldsymbol{u})\,\boldsymbol{v}^T}{1-\boldsymbol{v}^T\boldsymbol{u}}=$$

$$=\boldsymbol{I}-\boldsymbol{u}\boldsymbol{v}^T+\frac{1-\boldsymbol{v}^T\boldsymbol{u}}{1-\boldsymbol{v}^T\boldsymbol{u}}\boldsymbol{u}\boldsymbol{v}^T=\boldsymbol{I}.\checkmark$$

$$\boldsymbol{A}^{-1}\boldsymbol{A}=\left(\boldsymbol{I}+\frac{\boldsymbol{u}\boldsymbol{v}^T}{1-\boldsymbol{v}^T\boldsymbol{u}}\right)(\boldsymbol{I}-\boldsymbol{u}\boldsymbol{v}^T)=\boldsymbol{I}-\boldsymbol{u}\boldsymbol{v}^T+\frac{\boldsymbol{u}\boldsymbol{v}^T}{1-\boldsymbol{v}^T\boldsymbol{u}}-\frac{\boldsymbol{u}\,(\boldsymbol{v}^T\boldsymbol{u})\,\boldsymbol{v}^T}{1-\boldsymbol{v}^T\boldsymbol{u}}=\boldsymbol{I}$$

6. Find $\boldsymbol{A}^T, \boldsymbol{A}^{-1}, (\boldsymbol{A}^{-1})^T, (\boldsymbol{A}^T)^{-1}$ for

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix}.$$

Solution. By definition of transpose

$$\boldsymbol{A}^T = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}.$$

Apply Gauss-Jordan

$$[\,\boldsymbol{A} \;\; \boldsymbol{I}\,] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 9 & 3 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & -9 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1/3 \end{bmatrix} \Rightarrow \boldsymbol{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1/3 \end{bmatrix}.$$

$$(\boldsymbol{A}^{-1})^T = \begin{bmatrix} 1 & -3 \\ 0 & 1/3 \end{bmatrix} = (\boldsymbol{A}^T)^{-1}.$$

7. Describe within $\mathbb{R}^3$ the geometry of the column spaces of matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solution. $C(\boldsymbol{A})$ is a line since $\operatorname{rank}(\boldsymbol{A}) = 1$, $C(\boldsymbol{B})$ is a plane, $\operatorname{rank}(\boldsymbol{B}) = 2$, $C(\boldsymbol{A})$ is a line.

8. The vector subspaces of $\mathbb{R}^2$ are lines, $\mathbb{R}^2$ itself and $Z = \{[\,0 \;\; 0\,]^T\}$. What are the vector subspaces of $\mathbb{R}^3$?

Solution. Lines, planes, $\mathbb{R}^3$ itself and $Z = \{[\,0 \;\; 0 \;\; 0\,]^T\}$

9. Reduce the following matrices to row echelon form

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Solution. Obtain

$$\boldsymbol{A} \sim \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{B} \sim \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

10. Determine the null space of

$$\boldsymbol{A} = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix}.$$

Solution. By rref

$$\boldsymbol{A} \sim \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix},$$

hence $r = \operatorname{rank}(\boldsymbol{A}) = 1$, with $\boldsymbol{A} \in \mathbb{R}^{2\times 3} = \mathbb{R}^{m\times n}$. From FTLA $r + z = n$ with $z = \dim N(\boldsymbol{A}) = 3$. Two basis vectors for the null space can be chosen as

$$\boldsymbol{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}.$$