MATH347: Linear algebra for applications

Homework 11 - Solution

This assignment is a worksheet of exercises intended as preparation for the Final Examination. You should:

1. Review Lessons 13 to 24
2. Set aside 60 minutes to solve these exercises. Each exercise is meant to be solved within 3 minutes. If you cannot find a solution within 3 minutes, skip to the next one.
3. Check your answers in Matlab. Revisit theory for skipped or incorrectly answered exercies.
4. Turn in a PDF with your brief handwritten answers that specify your motivation, approach, calculations, answer. It is good practice to start all answers by briefly recounting the applicable definitions.

   When constructing a solution follow these steps:

a) Ask yourself: "what course concept is being verified?"
b) Identify relevant definitions and include them in your answer.
c) Briefly describe your approach
d) Carry out calculations
e) Present final answer

1 Matrix factorization

1. State $\boldsymbol{P} \in \mathbb{R}^{3\times 3}$ that permutes rows (1,2,3) of $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ as rows (2,3,1) through the product $\boldsymbol{PA}$.

   Solution.

2. Find the inverse of matrix $\boldsymbol{P}$ from Ex. 1.

   Solution.

3. State $\boldsymbol{Q} \in \mathbb{R}^{3\times 3}$ that permutes columns (1,2,3) of $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ as columns (3,1,2) through the product $\boldsymbol{AQ}$.

   Solution.

4. Find the inverse of marix $\boldsymbol{Q}$ from Ex. 3.

   Solution.

5. Find the LU factorization of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

   Solution.

6. Find the LU factorization of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

   Solution.

7. Prove that permutation matrices $\boldsymbol{P}, \boldsymbol{Q}$ from Ex.1,3 are orthogonal matrices.

   Solution.

8. Find the QR factorization of

$$\boldsymbol{A} = \begin{bmatrix} 0 & 5 & 6 \\ 0 & 0 & 9 \\ 1 & 2 & 3 \end{bmatrix}.$$

Solution.

9. Find the eigendecomposition of $\boldsymbol{R} \in \mathbb{R}^{2\times 2}$, the matrix of reflection across the first bisector (the $x = y$ line).

   Solution.

10. Find the SVD of $\boldsymbol{R} \in \mathbb{R}^{2\times 2}$, the rotation by angle θ matrix.

    Solution.

2 Linear algebra problems

1. Find the coordinates of $\boldsymbol{b} = [\ 6\ \ 15\ \ 24\ ]^T$ on the $\mathbb{R}^3$ basis vectors

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}.$$

   Solution.

2. Solve the least squares problem $\min_{\boldsymbol{x}} \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|$ for

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \boldsymbol{A} = \begin{bmatrix} 3 & -5 \\ -11 & 21 \\ 0 & 0 \end{bmatrix}.$$

   Solution.

3. Find the line passing closest to points $\mathcal{D} = \{(-2,3), (-1,1), (0,1), (1,3), (3,7)\}$.

   Solution.

4. Find an orthonormal basis for $C(\boldsymbol{A})$ where

$$\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}.$$

   Solution.

5. With $\boldsymbol{A}$ from Ex. 4 solve the least squares problem $\min_{\boldsymbol{x}} \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|$ where

$$\boldsymbol{b} = \begin{bmatrix} -4 \\ -3 \\ 3 \\ 0 \end{bmatrix}.$$

   Solution.

6. What is the best approximant $\boldsymbol{c} \in C(\boldsymbol{A})$ ($\boldsymbol{A}$ from Ex. 4) of $\boldsymbol{b}$ from Ex. 5?

   Solution.

7. Find the eigenvalues and eigenvectors of

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

   Solution.

8. For $\boldsymbol{A}$ from Ex. 7 find the eigenvalues and eigenvectors of $\boldsymbol{A}^2$, $\boldsymbol{A}^{-1}$, $\boldsymbol{A} + 2\boldsymbol{I}$.

   Solution.

9. Is the following matrix diagonalizable?

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution.

10. Find the SVD of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Solution.